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New York University
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Mathematics Research Group
Research Report No. EM-15

ASYMPTOTIC EVALUATION OF DIFFRACTIO[illegible]

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Title page
52 numbered pages

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ABSTRACT

The objectives of this paper are two-fold. In theory developed previously it is shown how to express any steady state electromagnetic field in the form of an asymptotic series. The basic variable in this series is the wave length, i.e., the first term is given for zero wave length. The present paper serves first as an illustration of how the theory on asymptotic expansions can be applied to physical problems. Secondly, the application itself is of interest and importance in the general problem of diffraction through lenses and as such it discusses a problem now actively pursued in current research. Insofar as application to diffraction is concerned this paper gives a wave formulation of a somewhat generalized problem of diffraction through lenses and obtains a solution of the formulation in terms of integrals. At this point the theory of asymptotic expansions is applied to evaluate the integrals. It turns out that the leading terms in the asymptotic expansions of the electromagnetic fields, i.e., the terms which represent the behavior of the field either for very small wave length at finite distances from the lens or for arbitrary wave length at infinite distances from the lens are precisely the terms given by geometrical optics. The higher order terms of the expansions, though not evaluated explicitly in this paper, give the diffraction effects which wave theory, as opposed to geometrical optics, encompasses.

1. Introduction

1.1. In an earlier paper the author showed how the complex spatial factors $\vec{u}$ and $\vec{v}$ of any steady state electromagnetic field can each be represented by an asymptotic series in which the basic variable is the wave-length[1] The coefficients of the terms in each asymptotic representation depend primarily upon the discontinuities with respect to time in what is called the pulse solution of Maxwell's equations. This solution is the field set up by charges which have the same geometrical distribution as the source of the steady state field but which are suddenly allowed to act.

The present paper shows how this general theory can be applied to the evaluation of integrals (see formulas (4.15) which arise in the study of diffraction of electromagnetic waves through a medium of variable electromagnetic parameters ϵ, μ, and σ. The mathematical formulation of the diffraction problem herein treated is a generalization of the common optical problem of diffraction through a lens. The formulation of the solution to the diffraction problem is not in itself especially novel. It leads in fact to a well known type of diffraction integral which is unfortunately too complicated to be readily evaluated except in the simplest of cases. The essence of this paper will then consist in showing how integrals of this type can be approximately evaluated by the use of the general theory of asymptotic expansions referred to above. It will be seen too that this method of evaluation is closely related to the stationary phase method as applied to integrals.

Unfortunately even the diffraction integrals herein presented as the solution to the problem of the diffraction caused by a medium of variable electromagnetic parameters are not exact solutions. Rather the diffracted field sought is required to satisfy some reasonable physical conditions and the integrals are shown to meet these conditions.

While one purpose of this paper is to illustrate the application of the general theory of asymptotic expansions, another is the development of the diffraction theory of optical aberrations. The physical meaning and

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[1] Luneberg, R.K.: Asymptotic Expansion of Steady State Electromagnetic Fields, Research Report No. EM-14, submitted by New York University to the Geophysical Research Directorate of the Air Force Cambridge Research Laboratories, AMC, U.S.Air Force. A more detailed review of this earlier theory will be given in article 2. Report No. EM-14 will be referred to as the first paper.

importance of such theory is discussed by Zerrnike.[2]

An analogous asymptotic evaluation of diffraction integrals is obtained by Van Kampen.[3] By using Kirchoff's principle Van Kampen expresses any component of the electric or magnetic field in the form of $\iint_D kg(x,y)e^{ikf(x,y)}dxdy$ where, in practical cases, D can be the domain of the aperture in a black screen provided D is bounded by one or more analytic curves. The asymptotic behavior of this integral for large k depends, Van Kampen finds, on the behavior of the integrand at three kinds of points; those at which the exponent is stationary, i.e.; $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$; those at which the exponent is stationary with respect to values on the boundary of D; and corner points, or points where the boundary of D fails to be analytic.

These same types of points appear in the theory of this paper, though the method of obtaining the asymptotic expansion is different. Van Kampen and the present author find that points of the second and third kind do not contribute to the zero order term in k. Hence their contribution represents diffractive effects as opposed to geometrical optics.

The diffraction study of this paper is closely related to numerous other studies of diffraction through apertures. For apertures in infinitely thin screens quite different methods have been employed[4] and, even when approximate, would seem to be more useful. For thick screens, however, the method of this paper provides an approach which can be employed if the geometrical optics solutions is known.

It should be noted that while the general theory of asymptotic development of steady state fields, as presented in the paper of footnote 1, is not restricted to fields in homogeneous media the application made to the diffraction theory in this paper is so restricted. The reason for this is that the integral formulation of the diffraction problem is obtained by the use of Green's theorem and since use of this theorem calls for knowledge of a suitable Green's function for the space of the diffracted field, present knowledge restricts the use of the theorem to homogeneous media.

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[2] Zerrnike, F.: Diffraction and Optical Image Formation, Physical Soc.,Proc., V. 61, 1948, pp. 158-164.

[3] Van Kampen, N.G.: An Asymptotic Treatment of Diffraction Problems, Physica, V. 14, Jan. 1949, pp. 575-589.

[4] See, for example, Levine, Harold and Schwinger, Julian: On the Theory of Diffraction by an Aperture in an Infinite Plane Screen.I. Physical Review, V.74, Oct.15, 1948,pp.958-974. The Introduction to this paper describes other methods of attack on the problem of diffraction through an aperture in an infinitely thin screen.

It is hoped that the general theory of asymptotic expansion of steady state fields will find application to other integrals arising in physically quite different problems which nevertheless lead to integrals such as arise in the diffraction problem. One such problem is that of electromagnetic propagation in non-homogeneous media. At the present time, however, successful application of the asymptotic theory to the integrals of the propagation problems has not been achieved.

1.2. Because the material of this paper is both lengthy and detailed a summary will be presented in this introduction.

The x,y,z space is divided into three domains, the domain $z < z_0$, called the object space, the domain $z > z_1$ called the image space and the domain $z_0 < z < z_1$ in which the material constants ϵ, μ, σ vary with x,y,z. We shall refer to this domain as the lens and to the entire object and image space and lens as the instrument. We assume that $\epsilon = n_0^2$ in the object and $\epsilon = n^2$ in the image space, n_0^2 and n^2 constant; we assume also that $\mu = 1$ and $\sigma = 0$ in object and image space. A time periodic point source is located at a point P_0 of the object space. The problem is to find the time periodic electromagnetic field.

E = Real part of $\vec{u}(x,y,z)e^{-i\omega t}$ and

H = Real part of $\vec{v}(x,y,z)e^{-i\omega t}$ which is established, in the image space, under the influence of this source. The vectors $\vec{u}$ and $\vec{v}$ are complex functions $a(x,y,z) + ib(x,y,z)$ and $c(x,y,z) + id(x,y,z)$ respectively. The vectors $\vec{u}$ and $\vec{v}$ then are solutions of the time free Maxwell's equations

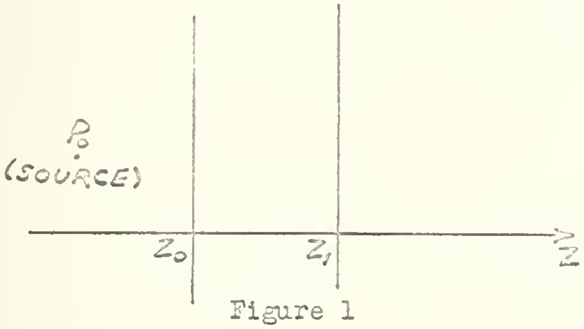

Figure 1

$$\begin{aligned} &\text{curl } \vec{v} + i\epsilon k\vec{u} = 0 \\ (1.1)\qquad &\text{curl } \vec{u} - i\mu k\vec{v} = 0 \end{aligned}$$

where $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$.

The solution $\vec{u},\vec{v}$ of our problem must be related to the geometical optics of our instrument, in particular to the system of light rays which originates at the point source P_0. This relation must be such that for small wavelengths, λ, that is, for large values of $k = \frac{2\pi}{\lambda}$, the vectors $\vec{u},\vec{v}$ show an asymptotic behavior which expresses the geometric optical situation created by the instrument. In other words we are concerned with a certain problem of correspondence: To find a solution $\vec{u},\vec{v}$ of (1.1) which, for $k \to \infty$, shows a prescribed asymptotic behavior.

We characterize the rays of the instrument by Hamilton's mixed characteristic $W(x_0,y_0,z_0\ ;\ p,q)$. This is a function of the coordinates x_0,y_0,z_0 of the source and of the direction numbers p,q,r of the rays in the image space. It determines the optical length of the ray between the point P_0 and the base point Q_0 of the perpendicular dropped from the origin 0 onto the ray p,q,r. We can determine all essential characteristics of the ray system with the aid of W.

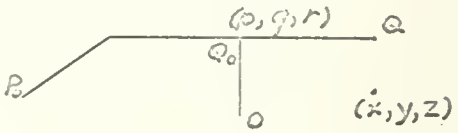

Figure 2

If x,y,z is any point in the image space and Q the base of the perpendicular from x,y,z to the ray with direction numbers p,q,r then the function

$$\phi(x,y,z) = px + qy + rz + W(p,q) \tag{1.2}$$

represents the optical length of the path from P_0 to Q, that is, the optical length $P_0\, Q_0$ plus the optical length Q_0Q. If now we determine the ray p,q,r which passes through a given point x,y,z, express p,q, and r in terms of x,y,z, and substitute these values of p,q, and r in ϕ we obtain the function $\psi(x,y,z)$ which gives the optical length from P_0 to (x,y,z). The surfaces $\psi(x,y,z) =$ constant are the wave fronts of the system, that is, the surfaces orthogonal to the rays.

There may, of course, exist several wave fronts through the same point x,y,z, just as there may exist several rays which connect this point with the source P_0.

Let $d\omega$ be the surface element in which a narrow "tube" of rays intersects a wave front ψ = constant. We measure the expansion of the ray system by the expression

$$K = \frac{dp\ dq}{d\omega\sqrt{1-p^2-q^2}}$$

which in our case of a homogeneous space is equal to the Gaussian curvature of the wave front. If the light distribution along a ray is observed then, according to geometrical optics, the polarization of light is unchanged whereas intensity and energy flux vary proportional to the above quantity K. We may describe these observations mathematically by a vector field $\vec{U}(x,y,z)$; the direction of these vectors determines the polarization of the light, the absolute value $|\vec{U}|^2$, the energy and flux. The vectors $\vec{U}$ along a fixed ray p,q,r are normal to the ray and are related by an equation $\vec{U} = \vec{A}(p,q)\sqrt{K}$ where the vector $\vec{A}(p,q)$ is constant along the ray.

The actual solution of the diffraction problem requires the integration of Maxwell's equations (1.1) where in the middle medium ϵ and μ are given functions of x,y,z. On account of the mathematical difficulties which oppose such an approach one prefers to establish solutions in the image space by making reasonable hypotheses:

1) The vectors $\vec{u}$ and $\vec{v}$ are analytical solutions of (1.1) which are regular and uniformly bounded in the total x,y,z space.

2) The vectors $\frac{\vec{u}}{\sqrt{K}}\ e^{-ik\psi}$ attain definite limit values $\vec{A}(p,q)$ and $\vec{B}(p,q)$ respectively, if infinity is approached along a ray p,q,r of the instrument. In other words it is assumed that at infinity both the actual electromagnetic field and the geometrical optics approximation have the same asymptotic behavior.

By application of Green's theorem it is shown that a solution satisfying these hypotheses can be obtained by superposition of plane waves, namely by the integrals

$$\vec{u} = -\frac{ik}{2\pi}\iint_D \frac{\vec{A}(p,q)}{\sqrt{n^2-p^2-q^2}}\ e^{ik\phi}\ dpdq$$

(1.3)

$$\vec{v} = -\frac{ik}{2\pi}\iint_D \frac{\vec{B}(p,q)}{\sqrt{n^2-p^2-q^2}}\ e^{ik\phi}\ dpdq$$

The domain of integration, D, is the range of all p and q values of rays which cover the image space and which are therefore determined by the medium between z_0 and z_1 or the lens. It is a domain of the p,q plane which lies inside the unit circle $p^2 + q^2 = n^2$. The integrals (1.3) are solutions of the equations (1.1) for any choice of vectors $\vec{A}$, $\vec{B}$ provided that the conditions

$$\vec{p} \times \vec{B} + \vec{A} = 0$$
$$\vec{p} \times \vec{A} - \vec{B} = 0 ,$$

wherein $\vec{p}$ is the vector (p,q,r), are satisfied.

Evaluation of the integrals (1.3) by known functions is possible only in a few simple cases. Thus we are led to the problem of determining the principal terms of these integrals by an asymptotic development with respect to a parameter proportional to the quantity $k = \frac{2\pi}{\lambda}$. The solution of this problem then should provide the answer to the question whether the integrals solve the above formulated problem of correspondence to a prescribed situation of geometrical optics. The answer will be in the affirmative: The principal terms of the integrals (1.3) are given by the expressions $\vec{u}_0 = \vec{A}\frac{\sqrt{K}}{k} e^{ik\psi}$, $\vec{v}_0 = \vec{B}\frac{\sqrt{K}}{k} e^{ik\psi}$.

We approach the problem of asymptotic development as follows: We introduce the vectors

$$\vec{P}(x,y,z,t) = \frac{1}{2\pi c} \iint_{\phi < ct} \frac{\vec{A}(p,q)}{\sqrt{n^2 - p^2 - q^2}} \, dpdq$$

(1.4)

$$\vec{Q}(x,y,z,t) = \frac{1}{2\pi c} \iint_{\phi < ct} \frac{\vec{B}(p,q)}{\sqrt{n^2 - p^2 - q^2}} \, dpdq$$

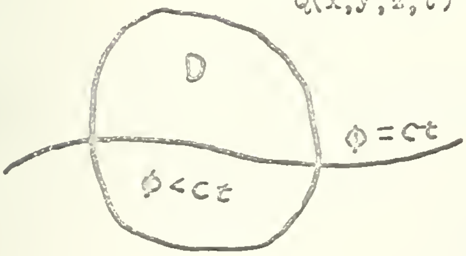

Figure 3

The domain of integration is that part of the domain D where

$$\phi = px + qy + rz + W < ct .$$

We then may write the multiple integrals in the form of simple integrals:

(1.5)
$$\vec{u} = -i\omega \int_0^\infty \vec{P}'(t) e^{i\omega t} \, dt$$
$$\vec{v} = -i\omega \int_0^\infty \vec{Q}'(t) e^{i\omega t} \, dt$$

The vectors $\vec{P}'$ and $\vec{Q}'$ are, at any fixed point x,y,z, different from zero only in a finite interval $T_1 < t < T_2$, determined by the extreme values of the function $\frac{1}{c}\phi$ in the domain D. In general $\vec{P}$ and $\vec{Q}$ will depend analytically on t; there are however certain points t_μ inside the above interval where $\vec{P}$ and $\vec{Q}$ or their derivatives are discontinuous. These points can be found by studying the topology of the level lines ϕ = const in D .

Consider for example a point x,y,z in the regular part of the field of rays (that is, only one ray passes through x,y,z). In this case ϕ possesses exactly one stationary point, P_0, in the interior of D; the associated value, ct_0, is either a maximum or minimum of ϕ and equals the optical $\psi(x,y,z)$ from the source to the point x,y,z. The level lines ϕ = constant thus surround the stationary point P_0 in the manner indicated in Figure 4. There exist two level lines

$$\phi = ct_1 \; , \qquad \phi = ct_2$$

(perhaps more) which are tangential to the boundary of D at points P_1 and P_2 . We can find these level lines by determining the stationary points of the boundary values of the function ϕ . If $\vec{A}$ and $\vec{B}$ are analytic in D then $\vec{P}'(t)$ and $\vec{Q}'(t)$ will be analytic functions of t except at the points

$$t_0 = \frac{1}{c}\psi_0 \quad ; \quad t_1 = \frac{1}{c}\psi_1 \quad ; \quad t_2 = \frac{1}{c}\psi_2$$

given by the points P_0, P_1, P_2 of Figure 4. The components of the vectors $\vec{P}$ and $\vec{Q}$ thus are functions of t of the type shown in Figure 5.

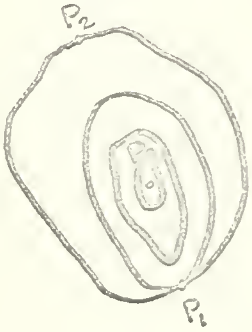

Figure 4

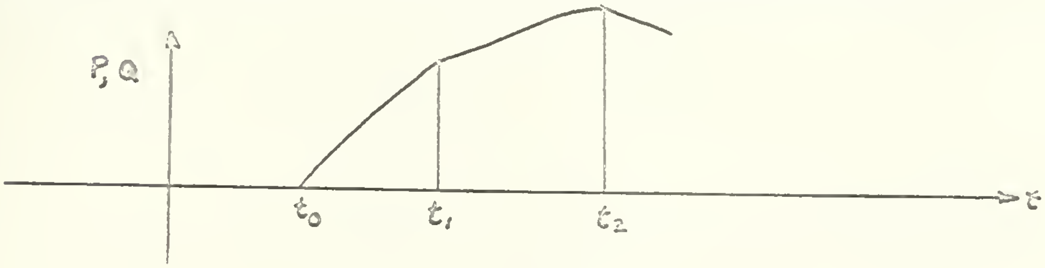

Figure 5

It is perhaps clear from the fact that $\vec{P}$ and $\vec{Q}$ will experience sudden changes in the rate of increase with t as t goes through the values t_o, t_1, and t_2 that these values of t will be points of discontinuity for $P'(t)$ and $Q'(t)$.

If x,y,z lies outside the field of the rays then there is no interior stationary point of ϕ but again two or more stationary boundary points P_1 and P_2. Hence there are only two points

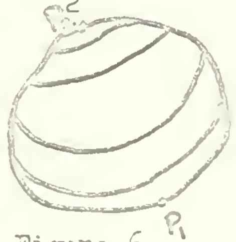

Figure 6

$$t_1 = \frac{1}{c}\psi_1(x,y,z) \; ; \quad t_2 = \frac{1}{c}\psi_2(x,y,z)$$

where $\vec{P}'$ and $\vec{Q}'$ are irregular.

If, finally, x,y,z lies in the irregular part of the ray system, that is, if more than one ray passes through x,y,z, then there will be more than one stationary point in the interior and a corresponding number of stationary boundary points. In this case there exist a certain finite number of points $t_\alpha = \frac{1}{c}\psi_\alpha(x,y,z)$ where $\vec{P}'$ and $\vec{Q}'$ are non-analytic.

Let us now assume that $\vec{P}$ and $\vec{Q}$ are sectionally analytic functions of t, the different branches being separated by a discrete, finite set of points

$$t_\alpha = \frac{1}{c}\psi_\alpha(x,y,z) \quad , \quad \alpha = 1, \ldots . N.$$

The points t_α are points of discontinuity of $\vec{P}'(t)$ and $\vec{Q}'(t)$. The representation (1.5) for $\vec{u}$ and $\vec{v}$ permits us to identify $\vec{P}'(t)$ and $\vec{Q}'(t)$ with the pulse solution $\vec{E}_o(t)$ and $\vec{H}_o(t)$ mentioned earlier and described in article 2. It is therefore possible to apply the results of the asymptotic theory presented in the earlier report. It follows from this theory that $\vec{u}$ and $\vec{v}$ admit asymptotic developments of the form

$$\vec{u} = \sum_{\alpha=1}^{N} \vec{U}_\alpha(x,y,z)e^{ik\psi_\alpha}$$

(1.6)

$$\vec{v} = \sum_{\alpha=1}^{N} \vec{V}_\alpha(x,y,z)e^{ik\psi_\alpha}$$

wherein the values of the vectors $\vec{U}_\alpha$ and $\vec{V}_\alpha$ depend upon the behavior of $\vec{E}_o(t)$ and $\vec{H}_o(t)$ in the immediate neighborhood of $t = t_\alpha = \frac{1}{c}\psi_\alpha(x,y,z)$.

If, for example, $\vec{E}_o(t)$ allows, in the neighborhood of $t = t_\alpha$, a representation of the following form

$$\begin{aligned} \vec{E}_o(t) &= (t_\alpha - t)^{\ell} \sum_{n=0}^{\infty} C_n (t_\alpha - t)^n , \quad t < t_\alpha \\ \vec{E}_o(t) &= (t - t_\alpha)^m \sum_{n=0}^{\infty} D_n (t - t_\alpha)^n , \quad t > t_\alpha , \end{aligned} \tag{1.7}$$

then the function U_α may be developed into an asymptotic series of the form

$$U_\alpha = \frac{c}{\omega} \sum_{n=0}^{\infty} \left[\frac{D_n \Gamma(n+1+m)}{(-i\omega)^{n+m}} - \frac{C_n \Gamma(n+1+\ell)}{(i\omega)^{n+\ell}} \right] \tag{1.8}$$

where $\Gamma(\chi)$ is the Gamma function.

The problem of asymptotic evaluation of the diffraction integrals is thus reduced to the problem of finding the character of the irregularities of the vectors $\vec{E}_o$ and $\vec{H}_o$ at certain singular points t_α. These singular points are determined by the system of the level lines ϕ = constant in D, in particular by the stationary points of the interior or of the boundary values of the phase function ϕ. We mention that an interior regular maximum or minimum point of ϕ introduces an ordinary discontinuity ($\ell = m = 0$) of $\vec{E}_o(t)$; a stationary boundary point, however, introduces a square root discontinuity ($\ell = 1/2$ or $m = 1/2$). This shows that the asymptotic contribution associated with stationary boundary points (P_1 and P_2 in Fig. 4) are of higher order in $\frac{1}{k}$ than those from interior stationary points. One also finds that the first coefficient C_o or D_o in case $\ell = m = 0$ has the value $\vec{A}\sqrt{K}$ so that the principal terms in the development (1.6) have the form

$$\vec{u}_o = \frac{\vec{A}}{k}\sqrt{K} \; e^{ik\psi} , \quad \vec{v}_o = \frac{\vec{B}}{k}\sqrt{K} \; e^{ik\psi} \tag{1.9}$$

as was stated above.

If the function ϕ possesses more than one interior stationary point then a finite sum of terms (1.9) is obtained as principal asymptotic contribution. This is the case if more than one ray passes through the point x,y,z in question.

2. Review of General Theory on Asymptotic Expansion of Steady State Fields.

This article will present without proofs the essential material of the theory of asymptotic expansions previously developed[5] and applied later in this paper.

By the *pulse solution* or *pulse field* satisfying Maxwell's equation we shall mean the functions $\vec{E}_0(x,y,z,t)$ and $\vec{H}_0(x,y,z,t)$ which are set up by a distribution of charge which at time $t = 0$ is suddenly allowed to act. The charge itself is characterized by a vector function $\vec{F} = \vec{g}(x,y,z)\, f(t)$ where $f(t)$ is 0 for $t \leq 0$ and $= 1$ for $t > 0$. The vector function $\vec{F}$ characterizes the charge in the sense that the current set up by enforced motion of the charge is given by $\vec{I} = \frac{1}{4\pi}\vec{F}_t(x,y,z,t)$ and the charge distribution itself is given by $\rho = -\operatorname{div}\vec{g}$. In the case $f(t)$ is the function just described, the effect of the sudden creation at $t = 0$ of the charge is to set up an electromagnetic field which, as $t \to \infty$, approaches the stationary electric field of a charge distribution $\rho = -\operatorname{div}\vec{g}$. That is $\vec{H}_0(x,y,z,t) \to 0$ as $t \to \infty$ for all x,y,z.

If the charge is characterized by a function $\vec{F} = \vec{g}(x,y,z)e^{-i\omega t}$ so that the enforced current is *periodic* while the charge distribution is $\rho = -\operatorname{div}\vec{g}$ then it is shown in the earlier paper that the field set up by the harmonic charge oscillations is, at $t = \infty$, $\vec{E}$ = Real part of $\vec{u}\,e^{-i\omega t}$ and $\vec{H}$ = Real part of $\vec{v}\,e^{-i\omega t}$ where

$$\vec{u}(x,y,z) = \vec{E}_0(x,y,z,\infty) - i\omega\int_0^{\infty}\left[\vec{E}_0(x,y,z,\tau) - \vec{E}_0(x,y,z,\infty)\right]e^{i\omega\tau}\,d\tau$$

(2.1)

$$\vec{v}(x,y,z) = \vec{H}_0(x,y,z,\infty) - i\omega\int_0^{\infty}\left[\vec{H}_0(x,y,z,\tau) - \vec{H}_0(x,y,z,\infty)\right]e^{i\omega\tau}\,d\tau$$

wherein the term $\vec{H}_0(x,y,z,\infty) = \vec{H}_0(\infty)$ has been retained for symmetry. The relation (2.1) shows that harmonic oscillations of charge produce at $t = \infty$ a harmonic steady state field.

From the purely mathematical relationship of $\vec{u}$ and $\vec{v}$ to the fields $\vec{E}_0(x,y,z,t)$ and $\vec{H}_0(x,y,z,t)$ as given by (2.1) it is possible to derive asymptotic formulas for $\vec{u}$ and $\vec{v}$ in terms of properties of the functions $\vec{E}_0$ and $\vec{H}_0$ and the circular frequency ω of the steady state field. One introduces the

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[5] See footnote 1.

vector fields A and B defined by

$$\vec{A}(t) = \frac{1}{2\pi i}\int_0^\infty \frac{\vec{E}_0(\tau)}{(\tau - t)^2}\, d\tau$$

(2.2)

$$\vec{B}(t) = \frac{1}{2\pi i}\int_0^\infty \frac{\vec{H}_0(\tau)}{(\tau - t)^2}\, d\tau$$

wherein the x,y,z dependence of $\vec{A}, \vec{B}, \vec{E}_0$, and $\vec{H}_0$ are understood and of course are fixed. Also, t is the complex variable $t_1 + it_2$. It is then shown that the singularities of $\vec{A}$ and $\vec{B}$ as functions of t are the singularities of $\vec{E}_0$ and $\vec{H}_0$, respectively, as functions of the real variable τ. The singularities of $\vec{E}_0$ and $\vec{H}_0$ occur at values $\tau_\alpha = \frac{1}{c}\psi_\alpha(x,y,z)$, that is, they occur at values of τ which lie on hypersurfaces $\psi_\alpha(x,y,z)$ and may arise physically from the sudden passage of wave-fronts across (x,y,z). These fronts themselves can arise from either the directly transmitted wave-front created by the sudden action of the charge as the front expands through space or from the passage of fronts reflected by or refracted through discontinuities in the whole space.

The basic result on the asymptotic form of $\vec{u}$ and $\vec{v}$ can now be stated. At a point x,y,z for which $\vec{g}(x,y,z) = 0$,

$$\vec{u} = \sum_{\alpha=1}^{\infty} \vec{U}_\alpha\, e^{ik\psi_\alpha}$$

(2.3)

$$\vec{v} = \sum_{\alpha=1}^{\infty} \vec{V}_\alpha\, e^{ik\psi_\alpha}$$

where

$$\vec{U}_\alpha = \int_M \vec{A}\left(\frac{\psi_\alpha}{c} + \tau\right) e^{i\omega\tau}\, d\tau$$

$$\vec{V}_\alpha = \int_M \vec{B}\left(\frac{\psi_\alpha}{c} + \tau\right) e^{i\omega\tau}\, d\tau$$

The path M surrounds the positive imaginary axis of the complex τ plane. Again the dependence of $\vec{u}, \vec{v}, \vec{U}_\alpha, \vec{V}_\alpha, \vec{A}$, and $\vec{B}$ on x,y,z is understood.

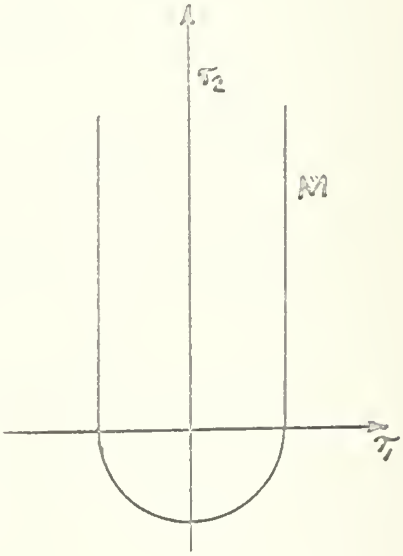

Figure 7

Since the values of $\vec{U}_\alpha$ and $\vec{V}_\alpha$ depend upon the singularities of the integrands $\vec{A}$ and $\vec{B}$ and since the singularities of $\vec{A}$ and $\vec{B}$ are those of $\vec{E}_0$ and $\vec{H}_0$ with respect to time, it is possible to determine a general result for the $\vec{U}_\alpha$ and $\vec{V}_\alpha$ by considering a general form for the singularity in $\vec{E}_0$ and $\vec{H}_0$. The general result is as follows: Let $t = t_\alpha$ $(= \frac{1}{c} \gamma_\alpha)$ be a discontinuity of $\vec{E}_0(t)$ and let $\vec{E}_0$ have the form

$$(2.4) \qquad \vec{E}_0(t) = \begin{cases} \vec{P}(t_\alpha - t) \displaystyle\sum_{n=0}^{\infty} \frac{\vec{C}_n}{n!} (t_\alpha - t)^n & , \text{ for } t < t_\alpha \\ \vec{Q}(t - t_\alpha) \displaystyle\sum_{n=0}^{\infty} \frac{\vec{D}_n}{n!} (t - t_\alpha)^n & , \text{ for } t > t_\alpha \end{cases}$$

In this expression $\vec{C}_n$ and $\vec{D}_n$ are constants. $\vec{P}$ and $\vec{Q}$ are analytic functions of $t_\alpha - t$ which, however, are singular at $t = t_\alpha$. For example $\vec{P}$ and $\vec{Q}$ may be of the form $(t - t_\alpha)^k$ where k is fractional or may be of the form $\log(t - t_\alpha)$.

If $\vec{E}_0(t)$ has the form given by (2.4) in the neighborhood of a singularity $t = t_\alpha$, $t_\alpha = \frac{\gamma_\alpha}{c}$, then $\vec{U}_\alpha$, which has the value $\int_M \vec{A}(t_\alpha + \tau) e^{i\omega\tau} d\tau$ is given by

$$(2.5) \qquad \vec{U}_\alpha = \sum_{n=0}^{\infty} \left\{ \frac{\vec{D}_n}{n!} \frac{1}{(-i\omega)^n} \int_0^\infty \vec{Q}\left(\frac{t}{-i\omega}\right) t^n e^{-t} \, dt - \frac{\vec{C}_n}{n!} \frac{1}{(i\omega)^n} \int_0^\infty \vec{P}\left(\frac{t}{i\omega}\right) t^n e^{-t} \, dt \right\}$$

The same result applied to $\vec{H}_0$ and $\vec{V}_\alpha$.

When one obtains the various $\vec{U}_\alpha$ and $\vec{V}_\alpha$ for all singularities $t_\alpha = \frac{\gamma_\alpha}{c}$, one has but to substitute in the summation formulas of (2.3) to obtain the asymptotic expressions for $\vec{u}$ and $\vec{v}$. The dependence of $\vec{u}$ and $\vec{v}$ on ω or on λ results through the expressions for $\vec{U}_\alpha$ and $\vec{V}_\alpha$ as may be seen in (2.5).

For many forms of the functions $\vec{P}$ and $\vec{Q}$ the integrals in $\vec{U}_\alpha$ are readily evaluated. For example, if $\vec{P}$ is of the form t^ℓ and $\vec{Q}$ of the form t^m where ℓ and m are greater than -1, then the integrals are readily evaluated through the Gamma function. In fact for these $\vec{P}$ and $\vec{Q}$,

$$(2.6) \qquad \vec{U}_\alpha = \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \frac{\vec{D}_n \Gamma(n+1+m)}{(-i\omega)^{n+m}} - \frac{\vec{C}_n \Gamma(n+1+\ell)}{(i\omega)^{n+\ell}} \right\}$$

The singularity in $\vec{E}_0$ at $t-t_\alpha$ may be a finite or infinite discontinuity, or $\hat{E}_0$ may be continuous with a discontinuity in one of its derivatives. It is apparent from (2.4) and (2.6) that in the expansion of $\vec{U}_\alpha$ the first term (that is for n=0) with $\omega \to \infty$ is the discontinuity in $\hat{E}_0$ at $t=t_\alpha$. (If $\hat{E}_0$ is continuous with some discontinuous derivative, the discontinuity in $\vec{E}_0$ is obviously 0). Hence for large ω, the sum of all such first terms would give the first terms in the expansions of $\hat{u}$ and $\hat{v}$, by virtue of formula (2.3). The latter terms would be $\sum [\vec{E}_0] e^{ik\psi_\alpha}$ and $\sum_\alpha [\vec{H}_0] e^{ik\psi_\alpha}$, where $[\vec{E}_0]$ denotes the values of the discontinuity in $\vec{E}_0$ at $t=t_\alpha$. The result apparent here from the somewhat specialized result holds generally[6].

3. The Diffraction Problem of Optical Instruments.

3.1. We formulate first the general diffraction problem which this paper treats. The problem is a generalisation of the usual physical situation created by placing a lens or an aperture in an infinite screen in the field of a source. In such a problem interest centers on finding the field transmitted through the lens, that is, the field in the image space.

An exact solution of the specific lens problem or of the generalization herein considered is not known. The problem is therefore modified somewhat in this article so that at least the reformulated problem can be attacked. The general theory of asymptotic evaluation of steady state fields is then applied to the new problem.

3.2. We define an optical instrument as a medium which is homogeneous in two half spaces $z < z_0$ and $z > z_1$ and which may have arbitrarily varying but sectionally continuous ϵ, μ, and σ in the domain $z_0 < z < z_1$. The domain $z_0 < z < z_1$ is referred to as the lens. Let us assume that ϵ, μ, σ have the constant values $\epsilon = n_0^2$, $\mu = 1$, $\sigma = 0$ in the "object space" and $\epsilon = n^2$, $\mu = 1$, $\sigma = 0$ in the "image space".

The field $\vec{E}$ and $\vec{H}$ established in the image space by a periodic source at P_0 will itself be periodic (see article 2) and will satisfy Maxwell's equations.

$$\begin{aligned} &\text{curl}\, \vec{H} - \frac{\epsilon}{c}\vec{E}_t - \frac{4\pi\sigma}{c}\vec{E} = 0 \\ (3.1)\quad & \\ &\text{curl}\, \vec{E} + \frac{\mu}{c}\vec{H}_t = 0 \end{aligned}$$

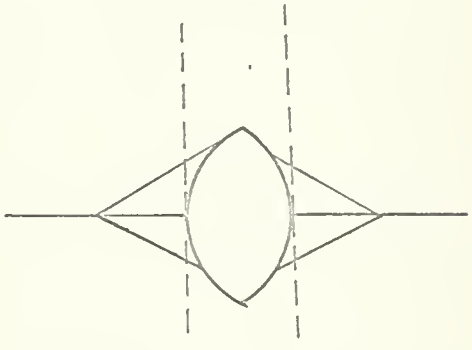

Figure 8

in source-free regions. If we write $\vec{E}$ and $\vec{H}$ as

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[6] See the report mentioned in footnote 1.

Real part of $\vec{u}(x,y,z)e^{-i\omega t}$ and Real part of $\vec{v}(x,y,z)e^{-i\omega t}$, respectively, where $\vec{u}$ and $\vec{v}$ are complex vectors, then $\vec{u}$ and $\vec{v}$ are themselves solutions of the time-free complex Maxwell equations

$$\text{curl}\,\vec{v} + ikn^2\vec{u} = 0 \qquad (3.2)$$
$$\text{curl}\,\vec{u} - ik\vec{v} = 0$$

We assume that there are no electric sources in the image space. It is assumed also that $\vec{u}$ and $\vec{v}$ are regular everywhere in the image space $z > z_1$.

The requirement that $\vec{u}$ and $\vec{v}$ are regular solutions of (3.2) is of considerable physical significance. According to geometrical optics an optical instrument may have a perfect focal point at a point P of the image space, i.e. a point towards which all rays from P_o converge. The assumption of regularity precludes a singularity even at P and hence implies the existence of a limit of resolution. No infinite energy densities will be possible at any point of the image space.

Let the time periodic source operating at P_o be characterized by $\vec{F} = \vec{g}(x,y,z)e^{-i\omega t}$ (see article 2), where $\vec{g} = 0$ with exception of a small neighborhood of P_o. We consider the associated "pulse source" $\vec{F} = \vec{g}(x,y,z)\eta(t)$ where $\eta(t)=0$ for $t < 0$ and $\eta(t)=1$ for $t > 0$ and assume that $\vec{E}_o(x,y,z,t)$ and $\vec{H}_o(x,y,z,t)$ represent the electromagnetic pulse propagated by this source. This field will be discontinuous on the branches of a hypersurface $\psi(x,y,z,t) = 0$, the branches being represented by $t = 1/c\,\psi_\alpha(x,y,z)$. In the image space where ϵ, μ, and σ are continuous (in fact, constant) and there are no sources it follows from our first paper that these branches are wave-fronts and so must satisfy the eiconal equation $\psi_x^2 + \psi_y^2 + \psi_z^2 = n^2$. These wave fronts characterize the internal reflections in the lens of the optical instrument.

If the functions ϵ, μ, σ of the optical instrument are sectionally analytic and if the source is given by a sectionally analytic vector field $\vec{F}$ then we can assume that the pulse solution (see article 2) $\vec{E}_o$ and $\vec{H}_o$ has sectionally analytic components. More specifically let us assume that the part of the x,y,z,t space which belongs to the image space $z > z_1$ is divided, by the hyper surfaces $\psi_\alpha = ct$, into domains in whose interior $\vec{E}_o$ and $\vec{H}_o$ possess derivatives of any order. Then by (2.3) we may represent the solution $\vec{u}$ and $\vec{v}$ of equations (3.2) by an infinite series of particular solutions of the form $\sum_\alpha \vec{U}_\alpha e^{ik\psi_\alpha}$ and $\sum_\alpha \vec{V}_\alpha e^{ik\psi_\alpha}$ respectively, where the summation over α is a summation over the many wave-fronts passing a given point x,y,z of space. As remarked in article 2, each of the $\vec{U}_\alpha$ and $\vec{V}_\alpha$ is an infinite series in the basic variable λ and the first terms in the infinite series for $\vec{u}$ and $\vec{v}$ are $\sum_\alpha [\vec{E}_o]_\alpha e^{ik\psi_\alpha}$ and $\sum_\alpha [\vec{H}_o]_\alpha e^{ik\psi_\alpha}$, respectively. These sums approximate $\vec{u}$ and $\vec{v}$ for small wave-lengths.

We consider in particular the part of each first term, namely $\vec{U}_0 e^{ik\psi}$ and $\vec{V}_0 e^{ik\psi}$ which belongs to the directly transmitted wave-front $\psi(x,y,z) = ct$, since it alone contributes to the image formation of the optical instrument. We assume that these two quantities are known, that is, we assume that the geometrical optics solution of our diffraction problem is known. This means that we know $[\vec{E}_0]\, e^{ik\psi}$ and $[\vec{H}_0] e^{ik\psi}$, where $[\vec{E}_0]$ and $[\vec{H}_0]$ are the electromagnetic vectors of the pulse solution on the extreme boundary of penetration of the transmitted wave-front $\psi(x,y,z) = ct$. Our problem thus becomes to *find a solution of Maxwell's equations (3.1) which in case $k \to \infty$ has this given asymptotic behavior*.

The $\vec{u}$ and $\vec{v}$ we seek are thus far not sufficiently limited to determine a unique solution. On the other hand the conditions presented by the presence of the medium between z_0 and z_1 are too difficult to treat exactly. We therefore rephrase our problem and thus the conditions on $\vec{u}$ and $\vec{v}$ to define a more amenable problem. The significance of the new conditions will be more apparent later on.

3.3. The wave fronts $\psi = ct$ of the directly transmitted wave are determined by the rays from the source at P_0. These rays form a two parameter system of straight lines in the image space, and the wave fronts $\psi = \text{const}$ are normal to these rays.

We now consider not only those straight line sections which lie in the half space $z > z_1$ but their total extension. This means that we supplement the image space by a "virtual" extension to a total homogeneous space. The rays of the optical instrument cover a certain part of the total image space which is called the *field of the instrument*. In particular we call the field *regular* in those parts where there is exactly one ray through each point. Obviously the field is not regular in the neighborhood of focal points or caustics. In fact two or more rays go through the same point.

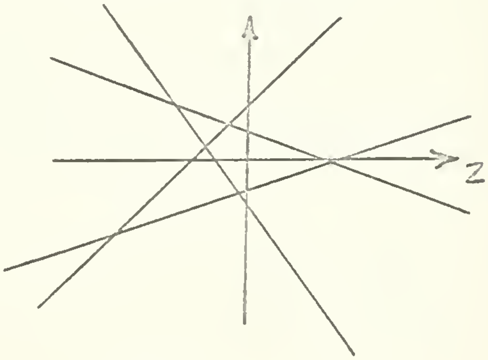

Figure 9

We shall confine our discussion to systems of rays which satisfy the following conditions:

1. The ray system is nowhere telescopic i.e. not more than one ray shall have the direction p, q, r. There shall be no "returning" rays i.e. $r \geqq 0$. A ray thus can be characterized uniquely by the coordinates p, q.

2. There exists a finite domain of the x, y, z space outside of which the rays form a regular field and the associated wave fronts satisfy the conditions: The Gaussian curvature, K, on any particular wave front is positive: $K > 0$. The wave fronts are convex towards the origin $x=y=z=0$ and $KR^2 \rightarrow 1$ if $R = \sqrt{x^2 + y^2 + z^2}$ approaches infinity along any one of the rays. This means the wave fronts assume, in the distant parts of the space, more and more the shape of spheres concentric to the origin.

3. There exist two wave fronts $\psi = C_1$ and $\psi = C_2$ on opposite sides of the origin which can be embodied into a closed convex surface, Σ, which has continuous surface normals and whose Gaussian curvature $K \geqq \delta > 0$. (Fig.10). The exterior normals of Σ determine, outside of Σ a regular field of straight lines which contains the rays of the instrument as a part. Since the rays are perpendicular to the wave-fronts, it is clear that any exterior parallel surface Σ_1 of Σ is a closed convex surface of the same type. Thus an infinite set of closed convex surfaces Σ exists which contain wave fronts of the instrument as parts. The two wave fronts which comprise part of Σ will be denoted by Σ' and Σ'', and Σ' will denote that section which lies in the field of the instrument.

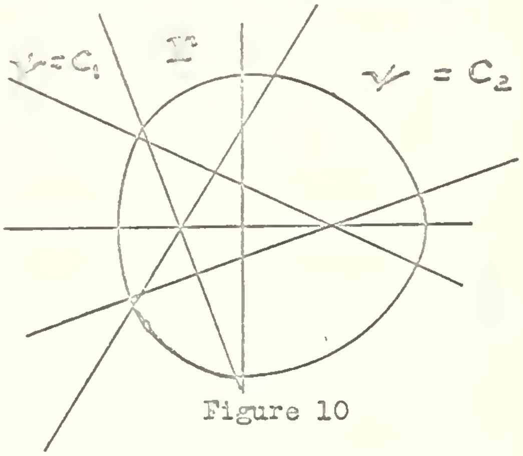

Figure 10

Consider now the vector fields $\vec{U}_0 = [\vec{E}_0]$ and $\vec{V}_0 = [\vec{H}_0]$ in a domain of the x, y, z space where the rays of the instrument form a regular field. As stated above, the quantities

$$\vec{U}_0 e^{ik\psi} \text{ and } \vec{V}_0 e^{ik\psi} \tag{3.3}$$

determine the electromagnetic vectors $\vec{u}$ and $\vec{v}$ of the directly transmitted field

in the case of small wave length. From the theory of the first paper[7] we know that in our homogeneous image space, $\vec{U}_0$ and $\vec{V}_0$ satisfy the conditions

$$(3.4)\qquad \begin{aligned} \operatorname{grad}\psi \times \vec{V}_0 + n^2\vec{U}_0 &= 0 \\ \operatorname{grad}\psi \times \vec{U}_0 - \vec{V}_0 &= 0 \end{aligned}$$

and the differential equations

$$(3.5)\qquad \begin{aligned} n\,\frac{d\vec{U}_0}{ds} + \frac{1}{2}\Delta\psi\,\vec{U}_0 &= 0 \\ n\,\frac{d\vec{V}_0}{ds} + \frac{1}{2}\Delta\psi\,\vec{V}_0 &= 0 \end{aligned}$$

along the rays. The parameter s measures the geometrical length of the rays. By using the relation (refer to (5.22) of the first paper and use the facts $\vec{p} = n^2\vec{s}$ and $\vec{p} = \operatorname{grad}\psi$)

$$(3.6)\qquad \Delta\psi = -n\,\frac{d\log|K|}{ds}$$

where K is the Gaussian curvature of the wave fronts[8] we may write (3.5) in the form

$$(3.7)\qquad \frac{d}{ds}\left(\frac{\vec{U}_0}{\sqrt{|K|}}\right) = \frac{d}{ds}\left(\frac{\vec{V}_0}{\sqrt{|K|}}\right) = 0$$

and conclude that the vectors

$$(3.8)\qquad \frac{\vec{U}_0}{\sqrt{|K|}} = \vec{A}(0,q)\ ;\quad \frac{\vec{V}_0}{\sqrt{|K|}} = \vec{B}(p,q)$$

are constant along a given ray p,q.

3.4. We use the relations (3.8) for the definition of a vector field $\vec{U}_0$, $\vec{V}_0$ in the whole image space. We consider the exterior normals of a surface Σ as defined in article 3.3. In the exterior of the surface Σ we define $\vec{U}_0$ and $\vec{V}_0$ by (3.9) if a point x,y,z lies in the field of the instrument and by $\vec{U}_0 = \vec{V}_0 = 0$ if not. This yields a sectionally continuous univalued vector field in the exterior of Σ.

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[7] See also Luneberg, R.K.: Mathematical Theory of Optics, Brown University Lectures, 1944, Vol. I, pp. 25 and 46.

[8] In the first paper K was introduced as a function of the ray parameter τ, and $K(\tau)$ measured the expansion of a wave-front along a ray. That $K(\tau)$ can be identified with the Gaussian curvature is shown by Silver, S.: Microwave Antenna Theory and Design, McGraw-Hill Book Co., N.Y., 1949, p.114, formula (22). For later use we note that the surfaces involved are arbitrary provided only that one considers expansion along a curve of the family of curves orthogonal to the surfaces.

We then consider the interior normals of Γ and transfer the vectors of the exterior field along these normals by requiring them to have the values given by (3.7), letting $\vec{A}(p,q) = \vec{B}(p,q) = 0$ if the normal does not belong to the rays of the instrument. The result is a multivalued vector field $\vec{U}_0, \vec{V}_0$ which has singularities at those points where $K = \infty$, that is, on the caustic of the ray bundle.

Since many wave-fronts may pass a given point x,y,z in space, in addition to the transmitted wave-front thus far emphasized we obtain more than one pair of vectors $\vec{U}_0$, $\vec{V}_0$ at a point through which more than one ray passes. In this case the function $\psi(x,y,z)$ is multivalued. If ψ', ψ'', ... are the values of ψ and $\vec{U}_0'$, $\vec{V}_0'$; $\vec{U}_0''$, $\vec{V}_0''$ the values of our vector field then

$$\begin{aligned} \vec{u} &\approx \vec{U}_0' \, e^{ik\psi'} + \vec{U}_0'' \, e^{ik\psi''} + \dots \\ \vec{v} &\approx \vec{V}_0' \, e^{ik\psi'} + \vec{V}_0'' \, e^{ik\psi''} + \dots \end{aligned} \tag{3.9}$$

give the asymptotic expression for the solution $\vec{u},\vec{v}$ in the case of small wave lengths in such a case.

We now consider Maxwell's equations (3.1) and (3.2) not only in the real image space $z > z_1$ but in the total homogeneous x,y,z space. We wish to find a solution $\vec{u},\vec{v}$ which in case of large values of k approaches asymptotically a given vector field of geometrical optics, namely the field (3.3) where $\vec{U}_0$ and $\vec{V}_0$ are given by (3.8) and $\vec{A}$ and $\vec{B}$ are known if the rays form a regular field, and the field (3.9) if not. We shall require in addition that $\vec{u}$ and $\vec{v}$ are regular and uniformly bounded in the whole x,y,z space and construct a solution of this problem in what is to follows.

It is physically plausible that the approximate solution of geometrical optics agrees the better with the actual steady state solution $\vec{u},\vec{v}$ the greater the distance of a point x,y,z from the irregular part of the field of the rays. Thus we are led to the conjecture that $\vec{u},\vec{v}$ attains the asymptotic form $\vec{U}_0 \, e^{ik\psi}$, $\vec{V}_0 \, e^{ik\psi}$ not only for large values of k at a fixed point x,y,z but also for large values of x,y,z and a fixed value of k. This behavior of $\vec{u}$ and $\vec{v}$ is indicated already by dimensional considerations. The degree of approximation must depend on a dimensionless quantity; we shall find $\frac{k}{\sqrt{K}}$ to be such a quantity. Indeed we know that the asymptotic development breaks

down at points on a caustic where $K = \infty$. Vice versa, the approximation for a given value of k will be best for smaller values of K, i.e., in the distant parts of the image space, for $K \to 0$ as our wave-fronts become spherical at large distances from the source.

We shall use the above hypothesis for the establishment of the solution of our problem in the form of complex integrals. Afterwards it will be shown that the hypothesis was justified by proving that the integrals have the desired asymptotic behavior at any fixed point x,y,z and for large values of k, as well as for fixed k and for x,y,z at large distances from the source.

3.5. The problem then is to construct a regular solution $\vec{u}, \vec{v}$ of the equations (3.2) which, at the infinity of the x,y,z space or for large k has a given asymptotic behavior:

$$\vec{u} - \vec{U}_0 \, e^{ik\psi} \to 0$$

$$\vec{v} - \vec{V}_0 \, e^{ik\psi} \to 0$$

where $\psi(x,y,z) = ct$ is the equation of the family of directly transmitted wave-fronts. By using the relations (3.8) we formulate this requirement in a more concise manner as follows: If ∞ is approached along a ray p,q of the instrument then there exists, to any $\epsilon > 0$, a sphere of radius R_0 around the origin such that

$$\left| \frac{\vec{u}\, e^{-ik\psi}}{\sqrt{K}} - \vec{A}(p,q) \right| \leq \epsilon \tag{3.10}$$

$$\left| \frac{\vec{v}\, e^{-ik\psi}}{\sqrt{K}} - \vec{B}(p,q) \right| \leq \epsilon$$

for all $R > R_0$.

If infinity is approached on a surface normal of a surface Σ which (normal) does not lie in the field of the instrument then

$$R\,|\vec{u}| \leq \epsilon \tag{3.11}$$

$$R\,|\vec{v}| \leq \epsilon$$

if $R > R_0$.

Finally we required that the solution $\vec{u}, \vec{v}$ of our problem satisfy a "radiation condition."

If ∞ is approached along a ray of the instrument or along any other surface normal of Σ then, if τ be the parameter along the ray or normal,

$$R\left|\frac{d\vec{u}}{d\tau} - ikn^2\,\vec{u}\right| \leq \epsilon$$

(3.12)

$$R\left|\frac{d\vec{v}}{d\tau} - ikn^2\,\vec{v}\right| \leq \epsilon$$

or explicitly[9]

$$R\left|\vec{u}_x\psi_x + \vec{u}_y\psi_y + \vec{u}_z\psi_z - ikn^2\,\vec{u}\right| \leq \epsilon$$

(3.13)

$$R\left|\vec{v}_x\psi_x + \vec{v}_y\psi_y + \vec{v}_z\psi_z - ikn^2\,\vec{v}\right| \leq \epsilon.$$

These last conditions are generalizations of the so-called Sommerfeld condition of radiation. They insure that $\vec{u},\vec{v}$ represent electromagnetic waves which have the same propagational character as a pulse traveling along the wave-fronts $\psi = ct$.

We shall therefore now seek a solution of the equations (3.2) which satisfies the conditions (3.10) to (3.13).

4. Integral Formulation of the Diffracted Field.

4.1. We now proceed to solve the reformulated diffraction problem. Briefly, the accomplishments of this article are as follows. In the homogeneous image space and the extension of it hypothesized in the preceding article, Maxwell's equations reduce to the wave equation. Green's theorem can then be employed to express $\vec{u}$ and $\vec{v}$ as integrals. Further analysis transforms the integrands so that they involve the functions $\vec{A}(p,q)$ and $\vec{B}(p,q)$ which are supposed known from a geometrical optics solution to the problem. The final integral formulation is given by (4.15).

Later articles will evaluate the integrals for $\vec{u}$ and $\vec{v}$ asymptotically and show that they have the expected behavior for large k or for large distances from the source.

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[9] If $\psi(x,y,z) = ct$ is equation of a family of wave fronts, and $x(\tau)$, $y(\tau)$, $z(\tau)$ the equations of an orthogonal trajectory, then ψ_x, ψ_y and ψ_z are proportional to x', y', and z'. Since $\psi_x^2 + \psi_y^2 + \psi_z^2 = n^2$, if we choose τ so that $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = n^2$, then $\psi_x = \dot{x}$, $\psi_y = \dot{y}$, $\psi_z = \dot{z}$. Hence

$$\frac{d\vec{u}}{d\tau} = u_x\,\psi_x + u_y\,\psi_y + u_z\,\psi_z\,.$$

4.2. To derive the integral representation for the vector fields $\vec{u}$ and $\vec{v}$ we eliminate $\vec{u}$ from the Maxwell equations in the usual way (3.2) by taking the curl of the first equation, substituting in the second, expanding curl curl $\vec{v}$ as $-\Delta\vec{v} + \nabla\nabla\cdot\vec{v}$ and using the fact that $\nabla\cdot\vec{v} = 0$. One obtains the wave equation for $\vec{v}$. Likewise one obtains the wave equation for $\vec{u}$. Specifically,

$$\vec{u} + k^2 n^2 \vec{u} = 0 \tag{4.1}$$
$$\vec{v} + k^2 n^2 \vec{v} = 0.$$

We now apply Green's theorem for the solution of equations (4.1). Consider the equation for $\vec{u}(x,y,z)$ and a domain D of the x,y,z space bounded by a closed surface Γ. We remove from G a small sphere of radius around a fixed point x_1,y_1,z_1 of D and apply Green's formula in the remaining part $D-\epsilon$ to the function $\vec{u}$ and the function $G = \frac{e^{inkR}}{R}$ where $R = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}$. Green's theorem states that

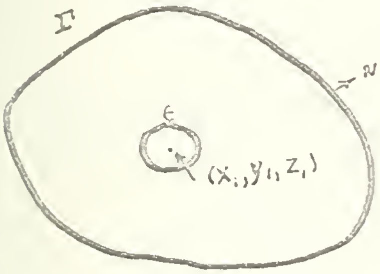

Figure 11

$$\iiint_{D-\epsilon} (\vec{u}\Delta G - \vec{G}\Delta u)\,dx\,dy\,dz = \iint_{\Gamma,\epsilon} \left(\vec{u}\,\frac{\partial G}{\partial\nu} - G\,\frac{\partial\vec{u}}{\partial\nu}\right) d\omega. \tag{4.2}$$

The surface integral is over the surfaces Γ and ϵ; the derivatives $\frac{\partial\vec{u}}{\partial\nu}$ and $\frac{\partial G}{\partial\nu}$ denote differentiation in the direction of the exterior normal of the boundary of $D-\epsilon$; and $d\omega$ is an element of surface.

It follows in the usual manner[10] letting $\epsilon\to 0$ that

$$4\pi\,\vec{u}(x_1,y_1,z_1) = -\iint_{\Gamma} \left(\vec{u}\,\frac{\partial G}{\partial\nu} - G\,\frac{\partial\vec{u}}{\partial\nu}\right) d\omega. \tag{4.3}$$

Since $\vec{u}$ is regular in the whole space we conclude that (4.3) holds for any closed surface Γ around the point x_1,y_1,z_1.

We can choose in particular any exterior parallel surface of distance ℓ from a fixed surface Γ_0 of the type assumed in article 3. Γ_0 as well as the parallel surface Γ contains two wave fronts ψ = const. Let us denote these sections by Γ_0', Γ_0'' and Γ', Γ'' respectively, and the remaining sections by Γ_0''' and Γ'''.

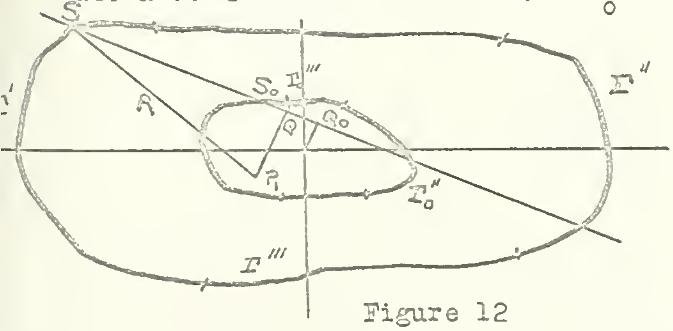

Figure 12

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[10] See, for example, Slater, J.C. and Frank, N.H.: Introduction to Theoretical Physics, McGraw-Hill Book Co., N. Y., 1933, pp. 218-219.

Since corresponding surface elements on Σ_o and Σ are related by the equation[11]

$$K d\omega = K_o d\omega_o$$

We can replace (4.3) by the integral over Σ_o :

(4.4) $$4\pi\vec{u} = -\iint_{\Sigma_o} (\vec{u}\ \frac{\partial G}{\partial \nu} - G\ \frac{\partial \vec{u}}{\partial \nu})\ \frac{K_o}{K}\ d\omega_o .$$

In this integral $\vec{u}$, G and K are the values assumed at the point S of Σ which lies on the surface normal through the point S_o .

Since (4.4) is true for all parallel surfaces Σ it remains true in the limit as $\ell \to \infty$, that is, we have

(4.5) $$4\pi\vec{u} = -\lim_{\ell\to\infty} \iint_{\Sigma_o} (\vec{u}\frac{\partial G}{\partial \nu} - G\ \frac{\partial \vec{u}}{\partial \nu})\ \frac{K_o}{K}\ d\omega_o$$

Hence the limit of the integral (4.5) may be found by determining the limit of the integrand provided that the convergence is uniform on Σ_o .

Now, since $G = \dfrac{e^{inkR}}{R}$, $\dfrac{\partial G}{\partial \nu} = (ikn - \dfrac{1}{R})\ G\ \dfrac{\partial R}{\partial \nu}$;

and since the wave fronts become more spherical in shape, $\frac{\partial R}{\partial \nu} \to 1$. Also, by an assumption of article 3, $KR^2 \to 1$. Further, the integrand in (4.5) thus has the same limit as the expression

(4.6) $$\left[\vec{u}(ikn - \frac{1}{R}) - \frac{\partial \vec{u}}{\partial \nu}\right] RK_o\ e^{iknR}.$$

But by assumption (3.10), since $K \to 0$ at infinity, $\vec{u} \to 0$. By assumption (3.12), if infinity is approached along a ray of the instrument

(4.7) $$R\left(\frac{d\vec{u}}{d\tau} - ik\ n^2 u\right) \to 0.$$

Since[12] $\dfrac{\partial \psi}{\partial \tau} = n^2$ we may write

$$R\left(\frac{d\vec{u}}{d\tau} - ik\ \frac{\partial \psi}{\partial \tau}\ \vec{u}\right) \to 0.$$

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[11] See footnote 8 of article 3.

[12] See footnote 9.

We may change parameters from τ to ν, the derivative in the direction of the normal, and cancel the factor $\frac{d\nu}{d\tau}$. However, (4.7) must be taken in the direction of a particular ray, (see the assumption in article 3.3 that the direction (p,q) is unique) whereas $\frac{\partial\psi}{\partial\tau}$ as required in the integral (4.5) must be outwardly drawn to Γ'. Hence we write

$$R\left(\frac{\partial\vec{u}}{\partial\nu} \pm ik\left|\frac{\partial\psi}{\partial\nu}\right|\vec{u}\right) \rightarrow 0. \tag{4.8}$$

Hence (4.6) has the same limit as

$$ik\left(n \pm \left|\frac{\partial\psi}{\partial\nu}\right|\right) R\,\vec{u}\,K_o\,e^{iknR} \tag{4.9}$$

for their difference approaches zero.

Now $\left|\frac{\partial\psi}{\partial\nu}\right| = |\mathrm{grad}\,\psi| = n$. If one chooses a ray normal to the portion Γ_o' of Γ_o along which to approach infinity then along this direction we must take the plus sign in (4.9), for we have denoted by Γ_o' that section of Γ_o which lies in the field of the instrument and from which rays of this field emerge. On Γ_o'', however, the direction of approach to infinity is the negative of the exterior normal's direction and so (4.9) vanishes on this segment of Γ_o. On Γ_o''' $R\vec{u}$ has the limit 0 by (3.11) and so (4.9) vanishes there also.

It follows then that the integral (4.5) need be taken over Γ_o' only and that the integrand may be taken to be

$$2\,i\,k\,n\,R\,\vec{u}\,K_o\,e^{iknR}\,. \tag{4.10}$$

It remains to determine the limit of $R\vec{u}\,e^{iknR}$. By (3.10) and the assumption that KR^2 approaches 1, we have that

$$R\vec{u} - \vec{A}(p,q)e^{ik\psi(S)} \rightarrow 0$$

as R approaches ∞.

Consider next the base point Q of the perpendicular dropped from (x_1,y_1,z_1) onto the ray through S_o (Figure 12). We have the relations $\psi(x,y,z)=ct$ for a wave front and $n = \frac{c}{v}$ where v is the velocity of an electromagnetic wave

along a ray in the medium of dielectric constant ϵ. Then $\psi(x,y,z)=nvt=n$ times the distance traversed by a point following an orthogonal trajectory to the family of wave fronts. In other words $\psi(S)$ is the optical length of the path from the source to S. Hence

$$\psi(S) = \psi(Q) + n(SQ)\ .$$[13]

But $SQ - R \to 0$, hence $\psi(S) - nR \to \psi(Q)$; that is, (4.10) has the limit $2i\,k\eta K_0\vec{A}(p,q)e^{ik\,\psi(Q)}$. We thus obtain for (4.5) the integral formula

$$u = -\frac{ikn}{2\pi}\iint_{\Gamma_0'} \vec{A}(p,q)e^{ik\,\psi(Q)}\,K_0\,d\omega_0$$

and similarly

$$\vec{v} = -\frac{ikn}{2\pi}\iint_{\Gamma_0'} \vec{B}(p,q)\,e^{ik\psi(Q)}\,K_0\,d\omega_0 \quad . \tag{4.11}$$

Since $K_0 d\omega_0 = Kd\omega$ is the same for all wave fronts it is clear that the integrals (4.11) are independent of the particular wave front Γ_0 which we have chosen.

4.3. As remarked above the function $\psi(Q)$ determines the length of the optical path along a ray p,q measured from the source P_0 to the base point Q of the perpendicular dropped from x_1,y_1,z_1 onto the ray. Consider now the base point Q_0 of the perpendicular dropped from the origin onto the ray.

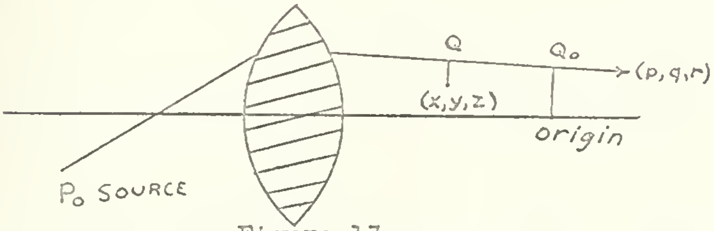

Figure 13

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[13] The quantity n(SQ) is the optical path from S to Q, the optical path being in general, by definition, $\int n\,ds$, the integral taken along a ray. Further $\int n\,ds = c\int\frac{ds}{v} = c\int dt$.

We shall use the fact taken from the theory of optics that the length of the optical path from P_o to Q_o is given by Hamilton's mixed characteristic of the optical instrument.[14] It is a function of the space coordinates of the point P_o and the direction numbers p,q,r of the ray in the image space and is denoted by

$$W(P_o,p,q) = W(p,q) = \psi(Q_o) .$$

Now since $\frac{p}{n}$, $\frac{q}{n}$ and $\frac{r}{n}$ are the direction cosines of the ray p,q,r, the optical path Q_oQ is given directly by $x_1p + y_1q + z_1r$.

Hence

$$\psi(Q) = W(p,q) + x_1p + y_1q + z_1r . \tag{4.12}$$

With the aid of W we thus may express the $\vec{u}$ and $\vec{v}$ of (4.11) in the form

$$\begin{aligned} \vec{u}(x,y,z) &= -\frac{ikn}{2\pi}\iint_{\Gamma_o'} \vec{A}(p,q)\, e^{ik(W+xp+yq+zr)}\, K_o d\omega_o \\ \vec{v}(x,y,z) &= -\frac{ikn}{2\pi}\iint_{\Gamma_o'} \vec{B}(p,q)\, e^{ik(W+xp+yq+zr)}\, K_o d\omega_o \end{aligned} \tag{4.13}$$

wherein we have dropped the subscripts from (x_1,y_1,z_1) .

Now the surface of integration can be any wave front of the instrument and the differential $Kd\omega$ is independent of the wave front. Hence we may determine it by choosing a wave front which is far away. Consider two wave fronts Γ_o' and Γ'. We have the relation $Kd\omega = K_o d\omega_o$ and, since, by assumption, $KR^2 \rightarrow 1$,

$$K_o d\omega_o = \lim_{R\rightarrow\infty} \frac{d\omega}{R^2}, \quad \text{where}$$

$$R = \sqrt{x^2+y^2+z^2} .$$

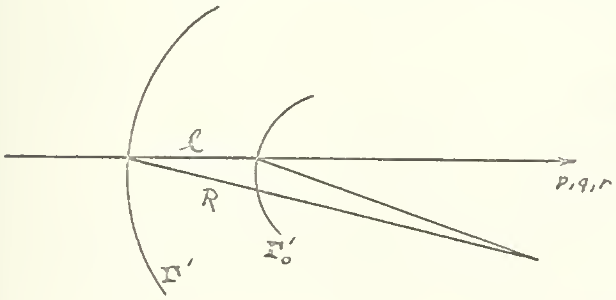

Figure 14

Now, let us assume that the wave front Γ_o' is given in parametric form

$$\vec{\zeta} = \vec{\zeta}\,(p,q) \tag{4.14}$$

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[14] See Synge, J.L.: Geometrical Optics, Cambridge University Press, 1937, p.24. or Luneberg, R. K.: Mathematical Theory of Optics, Brown University Lectures, p. 113.

that is, by a vector whose components are functions of p and q. The wave front Σ then is given by a vector

$$\vec{X} = \vec{\zeta}(p,q) + \frac{\ell}{n}\,\vec{p}$$

where ℓ is the constant distance of Σ from Σ_o and $\vec{p}$, the vector $(p,q,r) = p,q\sqrt{n^2-p^2-q^2}$. In accordance with a standard result of differential geometry the surface element $d\omega$ of Σ is given by the differential

$$d\omega = |\vec{X}_p \times \vec{X}_q|\,dpdq = |(\vec{\zeta}_p + \frac{\ell}{n}\vec{p}_p) \times (\vec{\zeta}_q + \frac{\ell}{n}\vec{p}_q)|\,dpdq\,.$$

However, $\frac{\ell^2}{R^2} \to 1$ if $R \to \infty$. Hence, it follows that

$$K_o\,d\omega_o = \lim_{R\to\infty} \frac{d\omega}{R^2} = \frac{1}{n^2}\,|\vec{p}_p \times \vec{p}_q|\,dpdq$$

$$= \frac{1}{n}\,\frac{dpdq}{\sqrt{n^2-p^2-q^2}}\,.$$

We introduce this expression in (4.13) and obtain the diffraction integrals in the final form

$$\vec{u}(x,y,z) = -\frac{ik}{2\pi}\iint_D \frac{\vec{A}(p,q)}{\sqrt{n^2-p^2-q^2}}\,e^{ik(W+px+qy+rz)}\,dpdq$$

(4.15)

$$\vec{v}(x,y,z) = -\frac{ik}{2\pi}\iint_D \frac{\vec{B}(p,q)}{\sqrt{n^2-p^2-q^2}}\,e^{ik(W+px+qy+rz)}\,dpdq.$$

The domain of integration D is the range of p,q values which belong to points on Σ_o' in accordance with (4.14). Now Σ_o' is the wave-front in the field of the instrument and hence is determined by the aperture of the instrument. It would consist then in the set of all p,q values of rays leaving the aperture of the instrument.

The integrals (4.15) are the basis of the diffraction theory of optical instruments. In the special case of a perfectly converging spherical wave we have W = const. In this case the integrals are known as Debye's solution of the wave equation.

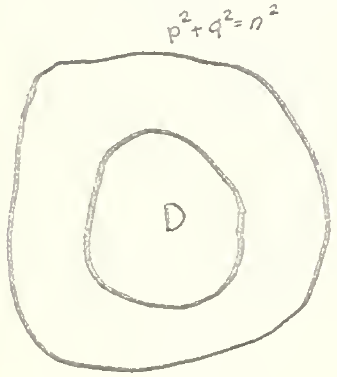

Figure 15

4.4. The integral representation (4.15) for $\vec{u}$ and $\vec{v}$ was arrived at by using some of the assumptions made in the preceding article, such as, for example, the regularity of $\vec{u}$ and $\vec{v}$ in the whole space and the asymptotic form at infinity. It remains to verify that these vector fields actually satisfy these conditions. We demonstrate first that $\vec{u}$ and $\vec{v}$ are regular solutions of Maxwell's equations. The proof that $\vec{u}$ and $\vec{v}$ have the required asymptotic behavior at infinity and beyond that the originally postulated behavior for large values of k will be given in the next two articles.

These integrals are no an exact solution of the diffraction problem. The assumptions introduced in article 3, which simplified the problem somewhat, have led to this result, and even though the integrals will prove to have the required asymptotic behavior, one must not conclude that the original problem was fully met. However, by virtue of the fact that the integrals do have the required asymptotic behavior they should serve at least as good approximations to the solution of the original problem.

It should be noted that the integrals can be evaluated directly in only a few simple cases. Hence the theory which follows on the asymptotic behavior of the integrals both at infinity and for large k values is practically a most useful approximate method of evaluation. As remarked elsewhere the results obtained may prove useful wherever integrals of this form arise.

The proof that $\vec{u}$ and $\vec{v}$ satisfy Maxwell's equations (3.2) is made as follows. By (3.4) and (3.8) it follows that $\vec{A}$ and $\vec{B}$ satisfy the condition

$$\begin{aligned} \vec{p}\times\vec{B}+n^2\vec{A}&=0 \\ \vec{p}\times\vec{A}-\vec{B}&=0 \end{aligned} \tag{4.16}$$

where $\vec{p}$ = grad ψ = (p,q,r).

Let us now form curl $\vec{u}$ and curl $\vec{v}$ by differentiating in (4.15) under the integral sign, remembering that $\vec{A}$ and $\vec{B}$ are independent of x,y,z. Then

$$\begin{aligned} \operatorname{curl} \vec{u} &= -\frac{(ik)^2}{2\pi} \iint_D \frac{\vec{p} \times \vec{A}}{\sqrt{n^2-p^2-q^2}} e^{ik(W+px+qy+rz)}\, dpdq \\ \operatorname{curl} \vec{v} &= -\frac{(ik)^2}{2\pi} \iint_D \frac{\vec{p} \times \vec{B}}{\sqrt{n^2-p^2-q^2}} e^{ik(W+px+qy+rz)}\, dpdq \ . \end{aligned} \tag{4.17}$$

Hence, on account of (4.16),

$$\begin{aligned} \operatorname{curl} \vec{v} + ikn^2\vec{u} &= 0 \\ \operatorname{curl} \vec{u} - ik\ \vec{v} &= 0 \ , \end{aligned} \tag{4.18}$$

that is, Maxwell's equations are satisfied by $\vec{u}$ and $\vec{v}$. It is clear that $\vec{u}$ and $\vec{v}$ are regular in the whole x,y,z space. Indeed we have the inequalities

$$\begin{aligned} |\vec{u}| &\leqq \frac{k}{2\pi} \iint_D \frac{|\vec{A}(p,q)|}{\sqrt{n^2-p^2-q^2}}\, dpdq \\ |\vec{v}| &\leqq \frac{k}{2\pi} \iint_D \frac{|\vec{B}(p,q)|}{\sqrt{n^2-p^2-q^2}}\, dp\, dq , \end{aligned} \tag{4.19}$$

that is, $\vec{u}$ and $\vec{v}$ are uniformly bounded in the whole x,y,z space.

5. Asymptotic Evaluation of the Diffraction Integrals for Large k.

5.1The integrals given by formulas (4.15) are the solution, then, to the diffraction problem as formulated in article 3. Except in simple cases it is not possible to evaluate them directly. Hence, one must seek other methods and, as already indicated, this paper will undertake asymptotic evaluation.

To obtain the asymptotic expansion of these diffraction integrals for large k values it is necessary to relate these integrals to the pulse solution discussed in article 2 and thereby to the general theory of asymptotic expansions reviewed there and more fully presented in the first paper. This

relation is established by means of a key step, the introduction of measure functions $\vec{P}$ and $\vec{Q}$, whose derivatives turn out to be $\vec{E}_0$ and $\vec{H}_0$. It is then necessary to obtain the discontinuities in $\vec{E}_0$ and $\vec{H}_0$.

5.2. The vector fields $\vec{u}$ and $\vec{v}$, as represented by the integrals (4.15), can be interpreted as a superposition of plane waves of various directions, amplitudes, and phases. The amplitudes are given by the expressions $\dfrac{\vec{A}(p,q)}{\sqrt{n^2-p^2-q^2}}$ and $\dfrac{\vec{B}(p,q)}{\sqrt{n^2-p^2-q^2}}$; the phase by $W(p,q)$, and the directions by the portion $px + qy + rz$ of the exponent. For convenience we introduce the notation

$$\phi(x,y,z,\ p,q) \equiv px + qy + rz + W(p,q)$$

For the asymptotic development of $\vec{u}$ and $\vec{v}$ the integrals (4.15) are transformed as follows. We introduce the "measure vectors"

$$\begin{aligned}\vec{P}(x,y,z,t) &= \frac{1}{2\pi c}\iint\limits_{\phi<ct} \frac{\vec{A}(p,q)}{\sqrt{n^2-p^2-q^2}}\,dpdq\\ \vec{Q}(x,y,z,t) &= \frac{1}{2\pi c}\iint\limits_{\phi<ct} \frac{\vec{B}(p,q)}{\sqrt{n^2-p^2-q^2}}\,dpdq\quad.\end{aligned}\tag{5.1}$$

For any fixed point x,y,z the integrals are to be extended over the part of the domain D of p and q values where

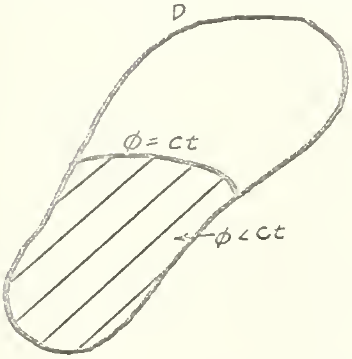

Figure 16

$$\phi = xp+yq+z\sqrt{n^2-p^2-q^2} + W(p,q) < ct.$$

If x,y,z is a point of the real image space then, in view of the physical meaning, ϕ is greater than 0.

We shall now show that the integrals (4.15) can be written[15] as simple Stieltjes integrals:

$$\begin{aligned}\vec{u} &= -\,i\omega\int_0^\infty e^{i\omega t}\,d\vec{P}(t)\\ \vec{v} &= -\,i\omega\int_0^\infty e^{i\omega t}\,d\vec{Q}(t).\end{aligned}\tag{5.2}$$

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[15] It should perhaps have been remarked earlier that all vector equalities are, as usual, no more than 3 separate scalar equalities, one for each the components. Thus the use of the Stieltjes integral with a vector integrand involves nothing new as compared with a scalar integrand.

The identification of (5.2) with (4.15) can be made as follows. For any fixed x,y,z, the function ϕ depends upon p and q. One may then consider the region D of p and q values to be broken up into sub-regions determined by $ct_{i-1} \leq \phi < ct_i$ where t_o, $t_1 \ldots t_{n-1}$, t_n are increasing values of t, and t_o and t_n (possibly infinite) are the extreme values of t which need be taken for a fixed x,y,z, to have the range of p and q values over which ϕ is defined cover the region D.

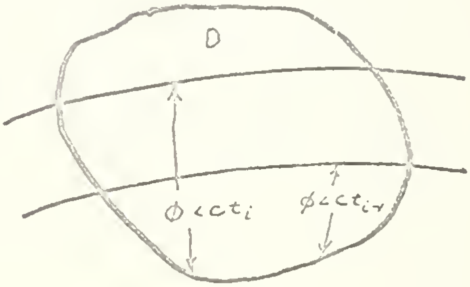

Figure 17

The choice of a value t_i of t determines by (5.1) the function $\vec{P}(t_i)$. Let us consider the expression

$$\left[\vec{P}(t_i) - \vec{P}(t_{i-1})\right] e^{ik\phi_{i-1}} , \tag{5.3}$$

where ϕ_{i-1} is the value of ϕ determined by any p_{i-1}, q_{i-1} in the region for which $ct_{i-1} \leq \phi < ct_i$. For definiteness p_{i-1}, q_{i-1} may be taken so that $\phi(p_{i-1}, q_{i-1}) = ct_{i-1}$. Now the expression (5.3), apart from a constant factor, is the value of one term in the sum which is formed to obtain $\vec{u}$ in (4.15). Indeed, regarded as a Stieltjes integral, the integral (4.15) is

$$\vec{u} = -2\pi c \frac{ik}{2\pi} \lim_{n\to\infty} \sum_{i=1}^{n} \left[\vec{P}(t_i) - \vec{P}(t_{i-1})\right] e^{ik\phi_{i-1}}$$

$$= -i\omega \lim_{n\to\infty} \sum_{i-1}^{n} \left[\vec{P}(t_i) - \vec{P}(t_{i-1})\right] e^{ik\,ct_{i-1}}$$

$$= -i\omega \int_0^{\infty} e^{i\omega t} \, d\vec{P}(t) .$$

The integration over the range 0 to co for t certainly yields the domain D of (4.15).

The identification of the form of $\vec{u}$ and $\vec{v}$ in (5.2) with the diffraction integrals (4.15) will now make possible the application of the general theory of asymptotic expansions.

We shall assume in the following that $\vec{A}(p,q)$, $\vec{B}(p,q)$ and $W(p,q)$ are integrable functions which are sectionally analytic. Also the boundary of D shall consist of a finite number of analytic curve sections. It is clear that then $\vec{P}$ and $\vec{Q}$ will be continuous, sectionally analytic functions of x,y,z,t. We may then replace $dP(t)$ by $P_t\,dt$. Also the time derivatives P_t and Q_t are sectionally analytic but not necessarily continuous. If they are integrable we introduce

$$\vec{E}_0(x,y,z,t) = \vec{P}_t$$
$$\vec{H}_0(x,y,z,t) = \vec{Q}_t \tag{5.4}$$

and may write

$$\vec{u} = -\,i\omega\int_0^\infty \vec{E}_0\, e^{i\omega t}\,dt$$
$$\vec{v} = -\,i\omega\int_0^\infty \vec{H}_0\, e^{i\omega t}\,dt\ . \tag{5.5}$$

Formulas (5.5) should now be compared with formulas (2.1) which relate the $\vec{u}$ and $\vec{v}$ factors of the steady state field with the pulse solution arising from the same source distribution. Actually (5.5) is a special case of (2.1) provided $\vec{E}_0$ and $\vec{H}_0$ are 0 at $t = \infty$. This is true for the vectors $\vec{E}_0$ and $\vec{H}_0$ defined by (5.4) and (5.1). In fact, from (5.1) it follows that for fixed x,y,z, $\vec{P}$ and $\vec{Q}$ are constant vectors in t for all t larger than a fixed value of t for which the set of points where $\phi < ct$ fills the total domain D. Hence $\vec{E}_0 = \vec{P}_t = 0$ and $\vec{H}_0 = \vec{Q}_t = 0$ for sufficiently large t.

It is also true that $\vec{P}$ and $\vec{Q}$ and hence the derivatives $\vec{E}_0 = \vec{P}_t$ and $\vec{H}_0 = \vec{Q}_t$ are solutions of Maxwell's equations. To prove this we introduce the functions

$$\chi(t - \tfrac{1}{c}\phi) = \begin{cases} \tfrac{1}{2}\,(t - \tfrac{1}{c}\phi)^2 & ;\quad t > \tfrac{1}{c}\,\phi \\ 0 & ;\quad t \leq \tfrac{1}{c}\,\phi \end{cases}$$

and write (5.1) in the form $\vec{P} = \dfrac{\partial^2}{\partial t^2}\vec{P}^*$; $\vec{Q} = \dfrac{\partial^2}{\partial t^2}\vec{Q}^*$ where

$$\vec{P}^* = \frac{1}{2\pi c}\iint_D \frac{\vec{A}(p,q)}{\sqrt{n^2 - p^2 - q^2}}\,\chi(t - \tfrac{1}{c}\,\phi)\,dpdq$$

$$\vec{Q}^* = \frac{1}{2\pi c}\iint_D \frac{\vec{B}(p,q)}{\sqrt{n^2 - p^2 - q^2}}\,\chi(t - \tfrac{1}{c}\,\phi)\,dpdq\ .$$

That $\vec{P} = \vec{P}^{*}_{tt}$ follows by differentiation under the integral sign and using the definition of $(t - \frac{\phi}{c})$.

We show first that $\vec{P}^{*}$ and $\vec{Q}^{*}$ satisfy Maxwell's equations. Since $(t - \frac{1}{c}\phi)$ has continuous derivatives, differentiations of $\vec{P}^{*}$ and $\vec{Q}^{*}$ with respect to x,y, and z can be carried out under the integral sign. Carrying out the differentiations called for in curl $\vec{P}^{*}$, rearranging terms, and using the fact that grad $\phi = (p,q,r) = \vec{p}$, we obtain

$$c \text{ curl } \vec{P}^{*} = -\frac{1}{2\pi c}\iint_D \frac{\vec{p} \times \vec{A}}{n^2 - p^2 - q^2}\, \chi'(t - \tfrac{1}{c}\phi)\, dpdq$$

$$\frac{\partial \vec{P}^{*}}{\partial t} = \frac{1}{2\pi c}\iint_D \frac{\vec{B}}{\sqrt{n^2 - p^2 - q^2}}\, (t - \tfrac{1}{c}\phi)\, dpdq$$

and two similar relations for c curl $\vec{Q}^{*}$ and $\frac{\partial \vec{Q}^{*}}{\partial t}$.

Hence by (4.16)

$$\begin{aligned} &\text{curl } \vec{Q}^{*} - \frac{n^2}{c}\vec{P}^{*}_{t} = 0 \\ (5.6)\qquad &\\ &\text{curl } \vec{P}^{*} + \frac{1}{c}\vec{Q}^{*}_{t} = 0 , \end{aligned}$$

that is, Maxwell's equations hold for $\vec{P}^{*}$ and $\vec{Q}^{*}$. By differentiating these equations twice with respect to t we obtain the same equations for $\vec{P}$ and $\vec{Q}$ and by a third differentiation Maxwell's equations for $\vec{E}_0$ and $\vec{H}_0$ everywhere where $\vec{E}_0$ and $\vec{H}_0$ are sectionally analytic.

We thus have found that the "measure vectors" $\vec{P}$ and $\vec{Q}$ and their derivatives $\vec{E}_0 = \vec{P}_t$, $\vec{H}_0 = \vec{Q}_t$ can be interpreted as pulse solutions of Maxwell's equations. As noted in article 2, the asymptotic behavior of the steady state solution $\vec{u},\vec{v}$ is determined by the discontinuities of the associated pulse solution $\vec{E}_0$, $\vec{H}_0$, that is, in our case by the discontinuities of the derivatives of $\vec{P},\vec{Q}$. We obtain the desired asymptotic development in the same manner as before. We introduce the vector fields

$$\begin{aligned} &\vec{M}(t) = \frac{1}{2\pi i}\int_0^\infty \frac{d\vec{P}(\tau)}{(\tau - t)^2} \\ (5.7)\qquad &\\ &\vec{\mathcal{N}}(t) = \frac{1}{2\pi i}\int_0^\infty \frac{d\vec{Q}(\tau)}{(\tau - t)^2} \end{aligned}$$

As pointed out in article 2 these vector fields are analytic functions of the complex parameter $t = t_1 + it_2$ and are regular in the whole t plane with exception of the positive real axis $t_1 \geqq 0$. The discontinuities of the derivatives of $\vec{P}$ and $\vec{Q}$ determine the location and the type of the singular points of $\vec{M}$ and $\vec{N}$ on the positive real axis.

By application of the theory reviewed in article 2, a representation of $\vec{u}$ and $\vec{v}$ by complex integrals can be obtained

$$\vec{u} = \int_L \vec{M}(t)\, e^{i\omega t}\, dt$$

(5.8)

$$\vec{v} = \int_L \vec{N}(t)\, e^{i\omega t}\, dt\ .$$

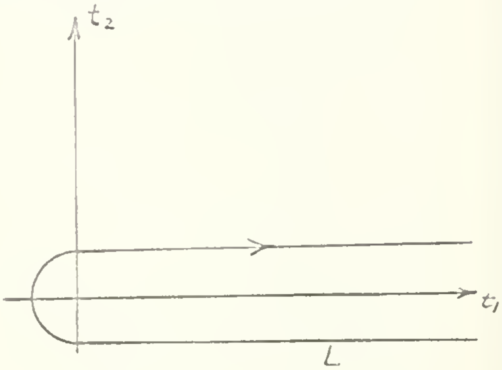

Figure 18

The points $t_\alpha = \frac{1}{c}\gamma_\alpha(x,y,z)$ where $\vec{P}(t)$ and $\vec{Q}(t)$ have discontinuous derivatives determine the branch points of the Riemannian surface of the vectors $\vec{M}$ and $\vec{N}$. Again in accordance with the theory of article 2, by deformation of the branch lines of this surface and of the path of integration we find the following representation of $\vec{u}$ and $\vec{v}$:

(5.9) $$\vec{u} = \sum_\alpha \vec{U}_\alpha\, e^{ik\gamma_\alpha}\ ;\quad \vec{v} = \sum_\alpha \vec{V}_\alpha\, e^{ik\gamma_\alpha}$$

where $\vec{U}_\alpha$ and $\vec{V}_\alpha$ are given by the integrals

$$\vec{U}_\alpha = \int_L \vec{M}\left(\frac{1}{c}\gamma_\alpha + \tau\right) e^{i\omega\tau}\, d\tau$$

(5.10)

$$\vec{V}_\alpha = \int_L \vec{N}\left(\frac{1}{c}\gamma_\alpha + \tau\right) e^{i\omega\tau}\, d\tau\ .$$

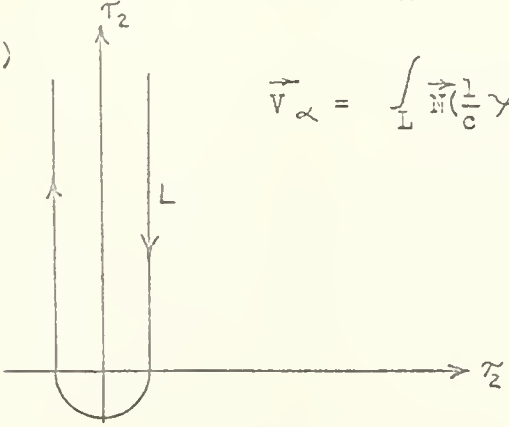

Figure 19

The path of integration, L, is now a curve of the complex $\tau = \tau_1 + i\tau_2$ plane which surrounds the positive imaginary axis.

We shall see now that $\vec{E}_0$ and $\vec{H}_0$ possess discontinuities of the type which will permit application of the general formula (2.5).

6. Analysis of the Discontinuities of $\vec{P}'(t)$ and $\vec{Q}'(t)$.

6.1. The method we shall use to determine the singularities of $\vec{P}'(t)$ and $\vec{Q}'(t)$ is analogous to the method of stationary phase. The analysis depends upon stationary values of the function $\phi = px + qy + rz + W(p,q)$ where $r = \sqrt{n^2 - p^2 - q^2}$. We shall therefore investigate the function ϕ somewhat.

In accordance with the facts introduced in article 4, the geometric meaning of ϕ is the optical length of the ray from the source P_0 to the base point, Q, of the perpendicular dropped from x,y,z onto the ray p,q.

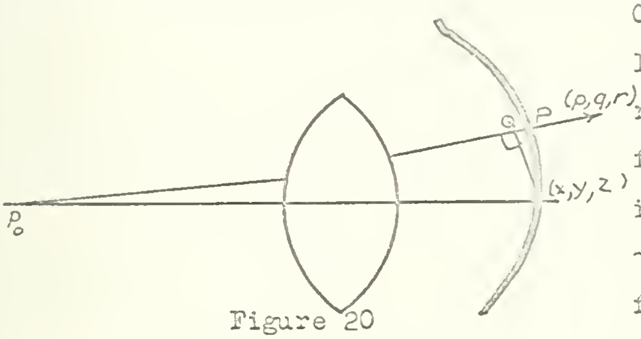

Figure 20

Consider now a fixed point x,y,z which lies in the regular part of the field of rays. Let $\psi(x,y,z) = C$ be the wave front through x,y,z, and let P be the intersection of the ray through Q with ψ. Since $\psi(P)$ is the optical path from P_0 to P, $\phi = C + n\,\overline{PQ}$. It is geometrically evident that $\overline{PQ}$ and hence ϕ attains a stationary value for a ray which passes through x,y,z. Obviously $\overline{PQ} = 0$ and thus $\underline{\phi = \psi(x,y,z)}$ on this particular ray.

We may thus determine the ray which goes through x,y,z by the conditions for a stationary value, namely,

$$(6.1) \qquad \begin{aligned} \phi_p &= x - \frac{p}{r} z + W_p = 0 \\ \phi_q &= y - \frac{q}{r} z + W_q = 0. \end{aligned}$$

If x,y,z lies in the regular field of the rays then a unique solution p(x,y,z), q(x,y,z) of (6.1) exists. However, the equations (6.1) are true also in the irregular part of the field. In this case several directions p(x,y,z), q(x,y,z) are obtained, that is, several rays pass through the point x,y,z.

Consider the planes

$$\phi(x,y,z;\ p,q) = px + qy + rz + W(p,q) = C. \tag{6.2}$$

These planes form a two parametric set depending upon the parameters p,q. Since $\phi_x^2 + \phi_y^2 + \phi_z^2 = n^2$ we recognize that $\phi(x,y,z;\ p,q)$ is a complete integral of the equation $\phi_x^2 + \phi_y^2 + \phi_z^2 = n^2$ and hence that (6.2) is a two parameter set of wave fronts. The envelope of the planes (6.2) thus must be again a wave front $\psi(x,y,z) = C$. We obtain $\psi(x,y,z)$ in the usual manner for determining the envelope of a family of surfaces, namely, by determining p and q from the equations (6.1) and by introducing the result in $\phi(x,y,z;\ p,q)$.

This function $\psi(x,y,z)$, of course, is the wave function of our system of rays. The set of planes $\phi(x,y,z;\ p,q) = C$ thus represents the tangential planes of the wave front $\psi(x,y,z) = C$.

We determine for a later application the Gaussian curvature of the wave front $\psi(x,y,z) = C$. We know that this surface can be found by eliminating p and q from the equations

$$\phi_p = 0, \quad \phi_q = 0, \quad \phi = C$$

or by (6.1)

$$\begin{aligned} x - \frac{p}{r} z + W_p &= 0 \\ y - \frac{q}{r} z + W_q &= 0 \\ px + qy + rz + W &= C. \end{aligned} \tag{6.3}$$

We also may use these equations to determine x,y,z as functions of p,q. This gives us the parameter representation of the wave front $\psi = C$:

$$\begin{aligned} x &= -W_p + \frac{p}{n^2}(C - W + p\,W_p + q\,W_q) \\ y &= -W_q + \frac{q}{n^2}(C - W + p\,W_p + q\,W_q) \\ z &= \frac{r}{n^2}(C - W + p\,W_q + q\,W_q)\ . \end{aligned} \tag{6.4}$$

The surface element $d\omega$ of the wave front $\psi = C$ is given by the expression

$$d\omega = |\vec{X}_p \times \vec{X}_q|\, dpdq$$

where $\vec{X}(p,q) = (x,y,z)$ is given by (6.3) or (6.4). By (6.3) it follows that

$$x_p - \frac{p}{r} z_p + \phi_{pp} = 0 \qquad x_q - \frac{p}{r} z_q + \phi_{pq} = 0$$

$$y_p - \frac{q}{r} z_p + \phi_{qq} = 0 \qquad y_q - \frac{q}{r} z_q + \phi_{qq} = 0$$

the quantity ϕ_{pp}, for example, being obtained through differentiation of (6.2) where ϕ is regarded as a function of the five independent variable x,y,z,p,q, and where $r = \sqrt{n^2 - p^2 - q^2}$. After an elementary calculation we find that

$$d\omega = \frac{r}{n}\left(\phi_{pp}\, \phi_{qq} - \phi_{pq}^2 \right) dpdq \;. \tag{6.5}$$

We now use the relation

$$Kd\,\omega = \frac{1}{n}\,\frac{dpdq}{r}$$

derived in article 4 and conclude that

$$K = \frac{1}{r^2(\phi_{pp}\;\phi_{qq} - \phi_{pq}^2)} \;. \tag{6.6}$$

This means: If the wave front $\psi(x,y,z) = C$ is characterized by the equations

$$\phi_p = 0, \quad \phi_q = 0 \qquad \phi(x,y,z;\ p,q) = C$$

then its Gaussian curvature K can be obtained with the aid of (6.6).

6.2. Consider first a point (x,y,z) which lies in the regular part of the field. More specifically let us consider points at the left side of the origin so that the wave fronts $\psi = ct$ are convex towards the right. We saw that at a fixed point, x,y,z, in the regular part of the field $\phi = px + qy + rz + W(p,q)$, regarded as a function of p and q, has exactly one stationary point in D and since this point must be a minimum point it follows that the contour lines $\phi(p,q)$ = constant in D must have the form indicated in Figure 21. These curves surround the minimum point C_o and touch the boundary of D at at least two points C_1 and C_2. We find the point C_o by the conditions $\phi_p = \phi_q = 0$. The associated minimum value of ϕ is given by the value of the function $\psi(x,y,z)$ at the point x,y,z.

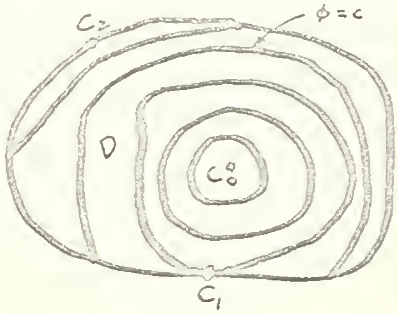

Figure 21

The points C_1 and C_2 also represent stationary values of $\phi(p,q)$ provided that the value of ϕ are considered on the boundary of D. Let $p = p(s)$, $q = q(s)$ be the parametric representation of the boundary of D. Then on the boundary of D $\phi(p,q) = \phi[p(r),q(s)]$. C_1 and C_2 may be found by the condition

$$\frac{d\phi}{ds} = \phi_p \frac{dp}{ds} + \phi_q \frac{dq}{ds} = 0. \tag{6.7}$$

The associated values $\phi = \psi_1(x,y,z)$; $\phi = \psi_2(x,y,z)$, considered as functions of x,y,z must be solutions of the equation $\psi_x^2 + \psi_y^2 + \psi_z^2 = n^2$; indeed these functions are defined by (6.7) as envelopes of a one parameter manifold of special solutions.

The vectors $\vec{P}$ and $\vec{Q}$ defined in (5.1) are sectionally analytic functions of t if we assume that $\vec{A}(p,q)$ is analytic and that the boundary of D is an analytic curve. Let us confine ourselves to this case. The only singular points of $\vec{P}(t)$ and $\vec{Q}(t)$ (i.e. points at which $\vec{P}$ and $\vec{Q}$ are not analytic though continuous) will be those of t which correspond to the points C_0, C_1, C_2 in Figure 21, that is, the values

$$t = \frac{1}{c}\psi\,; \quad t = \frac{1}{c}\psi_1; \quad t = \frac{1}{c}\psi_2 .$$

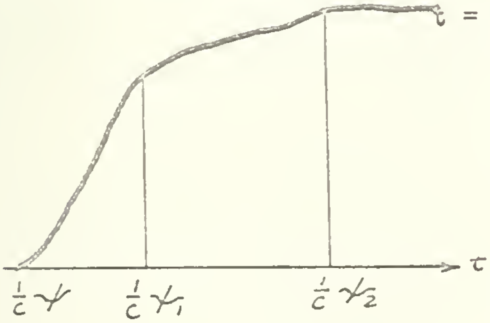

Figure 22

The components of $\vec{P}$ and $\vec{Q}$ thus are functions of t of the type illustrated in Figure 22. Since these values just found are the only ones in which $\vec{P}'$ and $\vec{Q}'$ are discontinuous our general result (5.9) allows us to conclude that $\vec{u}$ and $\vec{v}$ must have the form

$$\begin{aligned} \vec{u} &= \vec{U}\, e^{ik\psi} + \vec{U}_1\, e^{ik\psi_1} + \vec{U}_2\, e^{ik\psi_2} \\ \vec{v} &= \vec{V}\, e^{ik\psi} + \vec{V}_1\, e^{ik\psi_1} + \vec{V}_2\, e^{ik\psi_2} \end{aligned} \tag{6.8}$$

where $\vec{U}$, $\vec{V}$, $\vec{U}_1$, $\vec{V}_1$, $\vec{U}_2$, $\vec{V}_2$ are completely determined by the values of $\vec{P}$ and $\vec{Q}$ in the immediate neighborhood of the points $t = \frac{1}{c}\psi$, $t_1 = \frac{1}{c}\psi_1$, $t_2 = \frac{1}{c}\psi_2$, respectively, since these values of $\vec{P}$ and $\vec{Q}$ determine the singularities of $\vec{M}$ and $\vec{N}$ and therefore of the $\vec{U}_\alpha$ of (5.10).

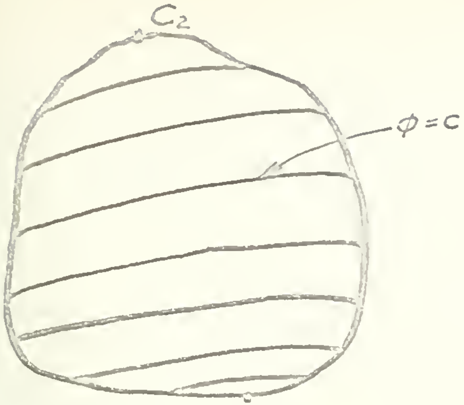

Figure 23

Consider next a point x,y,z outside the field of the instrument. Since no ray of the instrument passes through the point x,y,z it follows that the equations $\phi_p = \phi_q = 0$ have no solution inside D, that is, $\phi(p,q)$ has no stationary point in the interior of D. The contour lines ϕ = const. have the form illustrated in Figure 23. They touch the boundary at two points C_1 and C_2 in which the values of ϕ on the boundary assume extreme values. The associated values of t are $t_1 = \frac{1}{c}\psi_1$ and $t_2 = \frac{1}{c}\psi_2$. The components of $\vec{P}$ and $\vec{Q}$ have singularities (i.e. are non analytic) only at these particular values of t (Figure 24) and hence the representation

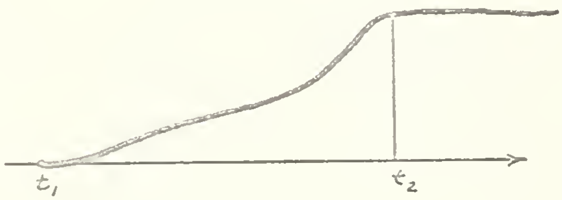

Figure 24

$$\vec{u} = \vec{U}_1 e^{ik\psi_1} + \vec{U}_2 e^{ik\psi_2}$$

(6.9)

$$\vec{v} = \vec{V}_1 e^{ik\psi_1} + \vec{V}_2 e^{ik\psi_2} .$$

If x,y,z lies to the right of the origin in the regular part of the field a similar pattern of the contour lines as shown in Figure 21 is obtained. The difference is that C_o is a maximum of ϕ. The components of P and Q depend on t as shown in Figure 25; $\vec{u}$ and $\vec{v}$ again have the form (6.8).

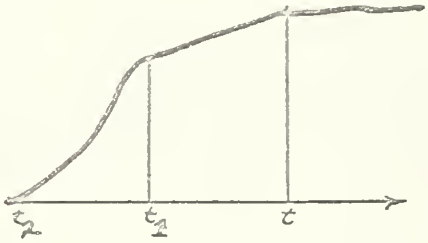

Figure 25

Consider finally a point x,y,z in the irregular part of the field. Since more than one ray passes through x,y,z it follows that the equations $\phi_p = \phi_q = 0$ have more than one solution, that is, there are several stationary points of ϕ in the interior of D. These stationary points can be minima, maxima, or saddle points (Figure 26). In addition to interior stationary points we obtain a number of stationary boundary points. The totality

of these points gives the set of points $t_\alpha = \frac{1}{c}\psi_\alpha$ where $\vec{P}(t)$ and $\vec{Q}(t)$ are not analytic. The vectors $\vec{u}$ and $\vec{v}$ are represented by series

$$\begin{aligned} \vec{u} &= \sum_{1}^{n} \vec{U}_\alpha \, e^{ik\psi_\alpha} \\ \vec{v} &= \sum_{1}^{n} \vec{V}_\alpha \, e^{ik\psi_\alpha} \end{aligned} \tag{6.10}$$

of finite many terms.

In what is to follow it will be assumed that x,y,z lies either outside the field or in its regular part. Thus we do not discuss, in general, the practically important question of the behavior of $\vec{u}$ and $\vec{v}$ in the neighborhood of a focal point.

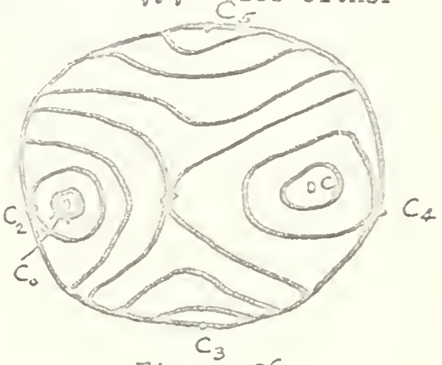

Figure 26

We also shall not abandon the assumption that the functions $\vec{A}(p,q)$, $\vec{B}(p,q)$ are analytic and that the boundary of D is an analytic curve. It is clear from the above that sharp corners of the boundary produce new singularities of $\vec{P}$ and $\vec{Q}$ and hence additional waves $\vec{U}_\alpha e^{ik\psi_\alpha}$, $\vec{V}_\alpha e^{ik\psi_\alpha}$ in the representation (6.8).

6.3. We now consider the singularities of $\vec{P}(t)$ and $\vec{Q}(t)$ at an interior stationary point. Let us assume that $\phi(p,q) = px + qy + rz + W(p,q)$ has a minimum at an interior point p_0, q_0 of the domain D. Let $ct_0 = \phi(p_0,q_0) = \psi(x,y,z)$ be the minimum value of ϕ. The vector $\vec{P}(t)$ is given by the integral (5.1), namely,

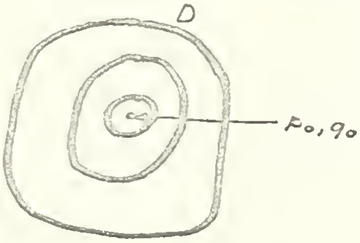

Figure 27

$$\vec{P}(t) = \frac{1}{2\pi c} \iint_{\phi \le ct} \frac{\vec{A}(p,q)}{r} \, dp\,dq \tag{6.11}$$

We now introduce, instead of p,q, the coordinates t, θ by the equations

$$\begin{aligned} p &= \rho(t,\theta) \cos\theta \\ q &= \rho(t,\theta) \sin\theta \end{aligned} \tag{6.12}$$

so that ρ, θ are polar coordinates around the point p_o, q_o. A contour line $\phi = ct$ is given by its equation in polar coordinates $\rho = \rho(t, \theta)$. For $t - t_o > 0$ sufficiently small a closed curve is obtained which is described once if θ goes from 0 to 2π. The components of the vector field $\frac{1}{r}\vec{A}(p,q)$ become functions of t and θ,

$$\frac{1}{r}\vec{A}(p,q) = \vec{A}^*(t,\theta) = \vec{A}^*(\rho\cos\theta,\ \rho\sin\theta)$$

In the neighborhood of the integral, (6.11), after a change of variable which requires the introduction of the Jacobian of the transformation, (6.12), becomes

(6.13) $$\vec{P}(t) = \frac{1}{2\pi c}\int_{t_o}^{t}\int_{0}^{2\pi} \vec{A}^*(t,\theta)\rho\rho_t \ dtd\theta .$$

We determine the function $\rho(t,\theta)$ from the equation

$$ct = \phi(p,q) = \phi(\rho\cos\theta, \rho\sin\theta).$$

We may expand $\phi(p,q)$ in a Taylor series in $\rho - \rho_o$ since p and q are functions of ρ. For example $\phi_\rho = \phi_p p_\rho + \phi_q q_\rho$, $\phi_{\rho\rho} = \phi_p p_{\rho\rho} + \phi_{pp} p_\rho^2 + \phi_{pq} q_\rho p_\rho + \ldots$

Also $\rho(t_o, \theta) = 0$ for $\rho(t,\theta)$ is the polar coordinate equation of $\phi = ct$ around t_o. Finally, $\phi_p = \phi_q = 0$ at $t = t_o$ for ϕ is stationary there. We have therefore in the neighborhood of $t = t_o$:

(6.14) $$c(t-t_o) = \rho^2\phi_2(\theta) + \rho^3\phi_3(\theta) + \ldots$$

where $\phi_2(\theta)$, $\phi_3(\theta)$, ... are homogeneous polynomials of order 2,3,.. in cos θ and sin θ. These polynomials satisfy the condition

(6.15) $$\phi_\nu(\theta + \pi) = (-1)^\nu \phi_\nu(\theta) .$$

We introduce the variable $s^2 = c(t-t_o)$ and consider ρ as function $\rho(s,\theta)$ of s and θ. By (6.14) it follows that

$$s = \rho\Phi(\rho,\theta)$$

where

$$\Phi(\rho,\theta) = \sqrt{\phi_2(\theta) + \rho\phi_3(\theta) + \ldots}$$

is a regular analytical function of ρ provided that

$$\phi_2(0) = \frac{1}{2}(\phi_{pp}\cos^2\theta + 2\ \phi_{pq}\cos\theta\sin\theta + \phi_{qq}\sin^2\theta) > 0.$$

This however follows from the fact that $\phi_{pp}\,\phi_{qq} - \phi_{pq}^{2} > 0$ at a minimum point of $\phi(p,q)$.

By (6.15) we conclude that $\bar{\Phi}(\rho,\theta)$ satisfies the identity

$$\bar{\Phi}(-\rho,\theta+\pi) = \bar{\Phi}(\rho,\theta) . \tag{6.16}$$

We also conclude that the equation $s = \rho\bar{\Phi}(\rho,\theta)$ can be inverted in a unique manner so that

$$\rho = sR(s,\theta)$$

where R is an analytical function of s.

The function $R(s,\theta)$ satisfies an identity similar to (6.16), namely, $R(-s,\theta+\pi) = R(s,\theta)$. In fact from $s = \rho\bar{\Phi}(\rho,\theta)$ it follows that $(-s) = (-\rho)\,\bar{\Phi}(-\rho,\theta+\pi)$ and, by the uniqueness of the inversion, $R(s,\theta) = R(-s,\theta+\pi)$.

We now introduce the following notation. A function $f(s,\theta)$ which depends only on even powers of s and satisfies the condition $f(s,\theta+\pi) = f(s,\theta)$ shall be denoted by $[G_0]$; a function $f(s,\theta)$ depending again only on even powers of s but satisfying the condition $f(s,\theta+\pi) = -f(s,\theta)$ shall be denoted by $[G_1]$. Obviously: $[G_0][G_0] = [G_0]$; $[G_1][G_1] = [G_0]$; $[G_0][G_1] = [G_1]$.

With this notation we can write

$$\rho = s\,[G_0] + [G_1]$$

and conclude readily

$$\rho_s = [G_0] + s\,[G_1]$$

and hence

$$\rho\rho_s = s\,[G_0] + [G_1] . \tag{6.17}$$

Consider next the function

$$\vec{A}^{*}(s,\theta) = \vec{A}^{*}\big(SR(s,\theta)\cos\theta,\ sR(s,\theta)\sin\theta\big) .$$

We conclude from $R(-s,\theta+\pi) = R(s,\theta)$ that $\vec{A}^{*}(s,\theta+\pi) = \vec{A}^{*}(s,\theta)$

$$\vec{A}^{*}(s,\theta) = [G_0] + s\,[G_1] . \tag{6.18}$$

By combining (6.17) and (6.18) we find

$$\vec{A}^{*}(s,\theta)\rho\rho_s = s[G_0] + [G_1] .$$

We introduce this result into the integral (6.13); since $\int_0^{2\pi}[G_1]\,d\theta = 0$ by the property of the G_1 function, we obtain

$$\vec{P}(t) = \frac{1}{2\pi c}\int_0^s s[G_0]\,ds = C_1 s^2 + C_2 s^4 + \dots ,$$

that is, $\vec{P}(t)$ depends only on even powers of $s = \sqrt{c(t-t_0)}$. This means $\vec{P}(t)$ is regular in the neighborhood $t \geq t_0$ of t_0.[16] A similar result, of course, will be found for the vector $\vec{Q}$.

In view of the regularity of $\vec{P}(t)$ for $t \geq t_0$ we may expand it and its derivative $\vec{E}_0$ (also $\vec{Q}$ and $\vec{H}_0$) in a Taylor's series, namely,

$$\vec{E}_0 = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}[\vec{E}_0]_\nu (t-t_0)^\nu \qquad \text{for } t \geq t_0 ,$$

(6.19)

$$\vec{H}_0 = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}[\vec{H}_0]_\nu (t-t_0)^\nu \qquad \text{for } t \geq t_0 ,$$

where $[\vec{E}_0]$ and $[\vec{E}_0]$ are the right hand derivatives of $\vec{E}_0$ and $\vec{H}_0$ at the point $t = t_0$, that is, the derivatives obtained by taking limits as t approaches t_0 from above. Let us determine explicitly the coefficients $[\vec{E}_0]_0$ and $[\vec{H}_0]_0$, that is, the discontinuities of the vector $\vec{E}_0$ and $\vec{H}_0$ at $t = t_0$. For very small values of t near t_0, $\vec{A}(p,q)$ does not vary much; hence we may say from (6.11) that

$$\vec{P}(t) \approx \frac{1}{2\pi c}\;\frac{\vec{A}(p_0,q_0)}{r_0}\iint_{\phi\leq ct} dp\,dq .$$

Since the Taylor's expansion of ϕ in the neighborhood of $t = t_0$ or (p_0,q_0) begins with second derivatives of ϕ with respect to p and q the integral on the right side is the area of the ellipse

$$\phi_{pp}(p-p_0)^2 + 2\,\phi_{pq}(p-p_0)(q-q_0) + \phi_{qq}(q-q_0)^2 = 2c(t-t_0) .$$

It follows that

$$\vec{P}(t) \approx (t-t_0)\,\frac{\vec{A}(p_0,q_0)}{r_0\sqrt{\phi_{pp}\,\phi_{qq} - \phi_{pq}^2}} ,$$

---

[16] $\vec{P}(t)$ is, of course, zero for $t < t_0$.

and hence with the aid of the result (6.10) :

$$(6.20)\qquad \begin{aligned}\left[\vec{E}_0\right] &= \vec{A}(p_0,q_0)\sqrt{K}\\ \left[\vec{H}_0\right] &= \vec{B}(p_0,q_0)\sqrt{K}\end{aligned}$$

where K is the Gaussian curvature of the wave front $\psi = ct_0$ through the point x,y,z in question.

The preceding result permist us to derivo the vector fields $\vec{U},\vec{V}$ in (6.8) . We apply the general formula (2.6) to our case, letting $m = \ell = 0$, $D_n = [E_0]_n$ and $C_n = 0$. It follows that

$$\vec{U} = \sum \frac{[\vec{E}_0]_n}{(-i\omega)^n} \quad ; \quad V = \sum \frac{[\vec{H}_0]_n}{(-i\omega)^n} \quad .$$

The first terms in these series are, in view of (6.20) ,

$$(6.22)\qquad \begin{aligned}\vec{U} \approx [\vec{E}_0]_0 &= \vec{A}(p_0,q_0)\sqrt{K}\\ V \approx [\vec{H}_0]_0 &= \vec{B}(p_0,q_0)\sqrt{K}\end{aligned}$$

We must now recall that if we fix on any point x,y,z in a regular part of the extended image space, there is one ray p_0,q_0 through x,y,z for which $\phi(p,q)$ is stationary. Also $\vec{A}(p_0,q_0)\sqrt{K}$, which is constant along this ray, gives the geometrical optics behavior of instrument. Hence the result: If p_0,q_0 is an interior stationary point of $\phi = px + py + rz + W(p,q)$ for a given x,y,z then the associated vector fields $U\,e^{ik\psi}$, $V\,e^{ik\psi}$ of the diffraction integrals approach, with increasing k, asymptotically the electromagnetic field of geometric optics, namely,

$$(6.23)\qquad \begin{aligned}\vec{U}_0\, e^{ik\psi} &= \vec{A}(p_0,q_0)\sqrt{K}\; e^{ik\psi}\\ \vec{V}_0\, e^{ik\psi} &= \vec{B}(p_0,q_0)\sqrt{K}\; e^{ik\psi}\end{aligned}$$

Since a unique p_0, q_0 exists and passes through any given point x,y,z in the regular part of the field (extended image space) the result applies for all rays in the regular part of the field.

The asymptotic behavior of the two other vector fields in (6.8) shall be determined in the next section. The result will be that they approach zero if $k \to \infty$.

6.4. We now consider the singularities of $\vec{P}(t)$ and $\vec{Q}(t)$ at stationary points of the boundary values of $\phi(x,y,z;\ p,q)$. Consider the case that $\phi = px + qy + rz + W(p,q)$ assumes a minimum value at an interior point C_0 of D, the domain of integration for the diffraction integrals, and that its boundary values on γ, the boundary of D, have a minimum at C_1 and a maximum at C_2 (Figures 21 and 28). The contour lines $\phi = ct_1$ and $\phi = ct_2$ touch the boundary γ at the points C_1 and C_2. Let us assume that $\phi_p^2 + \phi_q^2 \neq 0$ at these points, so that none of the points C_1 and C_2 is a real stationary point of ϕ with respect to values of ϕ in the interior of D. Let us also assume that the curvature at C_1 and C_2 of the tangential contour lines is different from the curvature of the boundary curve at these points.

The vector fields $\vec{A}(p,q)$ and $\vec{B}(p,q)$ shall be analytic functions of p and q defined in a domain D' somewhat larger than D.

In the neighborhood of $t = t_1$:

$$\vec{P}(t) = \frac{1}{2\pi c}\iint_{\phi\le ct} \frac{\vec{A}}{r}\,dpdq - \frac{1}{2\pi c}\iint_{D_1} \frac{\vec{A}}{r}\,dpdq\,. \tag{6.23a}$$

The first integral is extended over the domain in D' bounded by the closed contour line $\phi = ct$, the second integral over the sickle shaped domain D_1. The first integral is an analytic function of t, the second integral not because of the sharp corners in the boundary.[*] The singularities of $\vec{P}(t)$ thus are determined by the second integral.

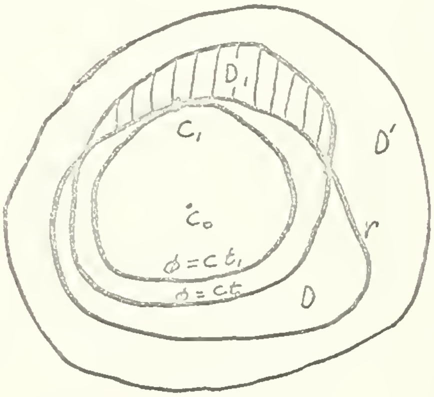

Figure 28

In the neighborhood of $t = t_2$, since ϕ has a maximum there with respect to values on γ,

$$\vec{P}(t) = \frac{1}{2\pi c}\iint_{D} \frac{\vec{A}}{r}\,dpdq - \frac{1}{2\pi c}\iint_{D_2} \frac{\vec{A}}{r}\,dpdq\,. \tag{6.23b}$$

The first integral is extended over the total domain D and thus is a constant. It follows that the second integral, which extends over the sickle shaped D_2,

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[*] Since, for $t < t_1$, the integral over D_1 is 0, and for $t > t_1$, the integral over D_1 is a non-constant function of t, the left-hand derivative of some order of this integral must be different from the right-hand derivative at $t = t_1$.

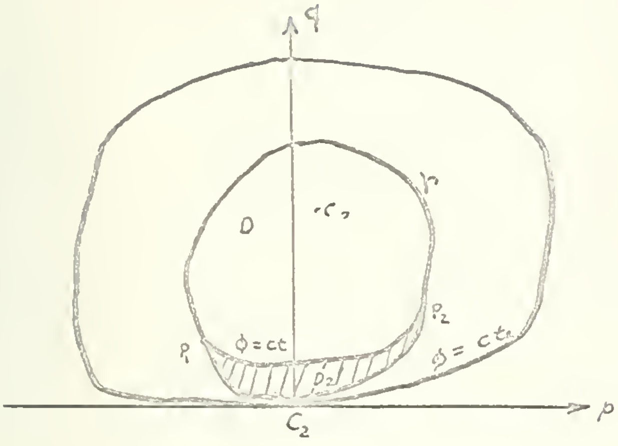

Figure 29

Figure 29, gives the singularities of $\vec{P}(t)$ in the neighborhood of $t = t_2$.

The type of singularity obviously will be the same at C_1 and C_2. Thus we may confine our discussion to the neighborhood of $t = t_2$.

We choose the coordinate system such that $p = q = 0$ at C_2 and that the boundary curve γ is tangential to the p-axis (Figure 29). Since $q' = \dfrac{dq}{dp} = 0$ at $p = 0$, $q = 0$ we may write

(6.24) $$q = \frac{K_0}{2} p^2 + q_3 p^3 + \ldots.$$

as the equation of the boundary curve; K_0 is its curvature at C_2.

The equation $q = q(p,t)$ of any contour line $\phi = ct$ may be obtained from solving $\phi = ct$, with t constant on the contour line, for q in terms of p. We may then consider the identity $\phi(p,q(p,t)) = ct$. For fixed t, $\dfrac{d\phi}{dp} = \phi_p + \phi_q \dfrac{dq}{dp} = 0$ since t is constant. Then since $\dfrac{dq}{dp} = 0$ at C_2 by virtue of the tangency of γ and the contour line, it follows that $\phi_p(o,o) = 0$. Since we have assumed that $\phi_p^2 + \phi_q^2 \neq 0$ at any stationary point of D, we have that $\phi_q(o,o) \neq 0$.

By expanding $ct = \phi(p,q)$ in a power series in p and q about $t = t_2$ or $p = 0$, $q = 0$, we obtain in the neighborhood of this point:

(6.25) $$\frac{c(t-t_2)}{\phi_q} = q + \frac{1}{2\phi_q}(\phi_{pp}p^2 + 2\phi_{pq}pq + \phi_{qq}q^2) + \ldots.$$

The contour line which touches the point C_2 is found from (6.25) by letting $t = t_2$; it has the form

(6.26) $$q = -\frac{\phi_{pp}}{2\phi_q} p^2 + \ldots.$$

so that $K = -\dfrac{\phi_{pp}}{\phi_q}$ is its curvature; our above assumption on the curvature of the contour lines means $K \neq K_0$, i.e., form (6.24) and (6.26) $K_0 + \dfrac{\phi_{pp}}{\phi_q} \neq 0$.

Now consider any contour line passing near but not necessarily through C_2. We determine the abscissae p_1 and p_2 of the points of intersection of this contour line $\phi = ct$ with the boundary γ. By inserting (6.24) into (6.25) we get

$$(6.27) \qquad \frac{2c}{\phi_q(K_o - K)}(t-t_2) = p^2(1 + a_1p + a_2p^2 + \ldots) \; .$$

Since the right side is positive for p small, an intersection is possible only for such values of $t - t_2$ which make the left side positive. Since C_2 is a maximum of ϕ on the boundary γ we have $\phi_q < 0$ near t_2 and hence $t < t_2$ if $K_o > K$ and $t > t_2$ if $K_o < K$. This result is geometrically obvious. We next introduce the quantity

$$(6.28) \qquad s^2 = \frac{2c}{\phi_q(K_o-K)}(t - t_2)$$

and use it in connection with formula (6.27) to calculate the abscissa p_o and p_2 from the equation

$$\pm s = p\sqrt{(1 + a_1p + a_2p^2 + \ldots)} \; .$$

It follows, since theoretically a solution for p in terms of s exists, that

$$(6.29) \qquad \begin{aligned} p_2 &= s\,R(s) \\ p_1 &= -s\,R(-s) \end{aligned}$$

where $R(s) = R_o + R_1s + \ldots$ is an analytic function of s. p_1 and p_2 then are the p values of the point of intersection of any contour line with γ.

As noted in connection with (6.23b), the singularities of p(t) at t_2 are given by the integral

$$(6.30) \qquad \vec{P}_2(t) = -\frac{1}{2\pi c}\iint_{D_2}\frac{\vec{A}}{r}\,dpdq = -\frac{1}{2\pi c}\int_0^s ds\int_{p_1}^{p_2}\frac{\vec{A}(p,s)}{r(p,s)}\,\frac{q}{s}\,dp.^{17}$$

Now, the equations (6.25) and (6.28) show that $q(s,p)$ is a function of s^2; it follows that $\vec{A}(p,s)$, which arises from $\vec{A}(p,q(s,p))$, and $r(p,s) = \sqrt{n^2 - p^2 - q^2(s,p)}$ are also functions of s^2. $\frac{\partial q}{\partial s}$ thus is s times a function of s^2. The second integral in (6.30) then has the form

$$\vec{P}_2(t) = + \frac{1}{2\pi c}\int_0^s sds\int_{p_1}^{p_2}(\vec{\sigma}_o(p) + \vec{\sigma}_1(p)s^2 + \vec{\sigma}_2(p)s^4 + \ldots)\,dp.$$

---

[17] The second integral may be obtained from the first by the change of variables $p = p$, $q = q(s)$. The Jacobian of the transformation is then $\frac{\partial q}{\partial s}$.

We conclude from (6.29) that the integral $\int_{p_1}^{p_2} \vec{\partial}(p)dp$ must be an odd function of s and hence, that $\vec{P}_2(t)$ has the form

$$\vec{P}_2(t) = + \int_0^s s^2\,(\vec{C}_o + \vec{C}_1 s^2 + \ldots)\,ds = s^3(\vec{C}'_o + \vec{C}'_1 s^2 + \vec{C}'_2 s^4 + \ldots)$$

By reintroducing the variable t by means of (6.28) it is found that

$$\vec{P}_2(t) = + \sqrt{t_2-t}^{\,3}\,(\vec{C}_o + \vec{C}_1(t_2-t) + \vec{C}_2(t_2-t)^2 + \ldots)\ . \tag{6.31}$$

This gives for the singular parts of $\vec{E}_o = \vec{P}_t(t)$ and similarly of $\vec{H}_o = \vec{Q}_t$ expressions of the type:

$$\begin{aligned}\vec{E}_o &= + \sqrt{t_2-t}\,\Big(\sum \frac{\vec{C}_\nu}{\nu!}(t_2-t)^\nu\Big) + \text{Reg. functions}\\ \vec{H}_o &= + \sqrt{t_2-t}\,\Big(\sum \frac{\vec{D}_\nu}{\nu!}(t_2-t)^\nu\Big) + \text{Reg. functions.}\end{aligned} \tag{6.32}$$

Let us determine the first coefficients $\vec{C}_o$ and $\vec{D}_o$ of these expressions. From (6.25) and (6.28) we get

$q = - \dfrac{K-K_o}{2} s^2 + \ldots$. From (6.29) we have $p_1 = - s + \ldots$ and $p_2 = s + \ldots$. Hence, since $\vec{P}_2(t) = \dfrac{1}{2\pi c}\displaystyle\int_0^s ds \int_{p_1}^{p_2} \frac{\vec{A}}{r}\frac{\partial q}{\partial s}dp$, and since in the neighborhood of C_2 we may regard $\dfrac{\vec{A}}{r}$ as constant and hence remove it from the integral,

$$\vec{P}_2(t) = - \frac{K-K_o}{2\pi c}\,\frac{\vec{A}}{r}\int_0^s s\,ds \int_{-s}^{s} dp + \ldots = - \frac{K-K_o}{3\pi c}\,\frac{\vec{A}}{r}\,s^3 + \ldots .$$

From (6.28) we then obtain

$$\vec{P}_2(t) = - \frac{K-K_o}{3\pi c}\,\frac{\vec{A}}{r}\left(\frac{2c(t-t_2)}{\phi_q(K_o-K)}\right)^{3/2} + \ldots\ .$$

In a coordinate system not specialized as in Figure 29 one has to replace ϕ_q by $\sqrt{\phi_p^2 + \phi_q^2}$. Hence it follows by differentiation of $\vec{P}_2(t)$ (and similarly of $\vec{Q}_2(t)$) and by reference to (6.32) that

$$\begin{aligned}\vec{C}_o &= \frac{1}{\pi}\frac{\vec{A}}{r}\sqrt{\frac{2c}{|K-K_o|}}\,\frac{1}{(\sqrt{\phi_p^2 + \phi_q^2})^{3/2}}\\ \vec{D}_o &= \frac{1}{\pi}\frac{\vec{B}}{r}\sqrt{\frac{2c}{|K-K_o|}}\,\frac{1}{(\sqrt{\phi_p^2 + \phi_q^2})^{3/2}}\ .\end{aligned} \tag{6.33}$$

Our result is not valid if $\phi_p^2 + \phi_q^2 = 0$ i.e. if C_2 is a real stationary point of ϕ located on the boundary γ; it also breaks down if $K = K_0$ i.e. if the tangential contour line $\phi = ct_2$ and the boundary curve γ have, at C_2, the same curvature. These exceptional cases require a special consideration.

We use the above results to derive the remaining vector fields $\vec{U}_1$, $\vec{V}_1$; $\vec{U}_2$, $\vec{V}_2$ in (6.8). We know from (6.32) that at $t = t_2$

$$\text{(6.34)} \qquad \begin{aligned} \vec{E}_0 &= \sqrt{t_2-t} \sum_{\nu} \vec{C}_{\nu}^{(2)}/\nu! \; (t_2-t)^{\nu} && \text{for } t < t_2 \\ &= 0 && \text{for } t > t_2 \; . \end{aligned}$$

The zero value of $\vec{E}_0$ for $t > t_2$ holds because t_2 is a maximum relative to values of ϕ on γ and so is all the more a maximum for ϕ for values assumed by all p,q in D. (See Figure 29). Hence $\vec{P}(t) = \iint_{\phi < ct} \frac{\vec{A}}{n}\, dpdq$ is constant for $t > t_2$ and $\vec{E}_0(t) = \vec{P}_t = 0$ for $t > t_2$.

In the neighborhood of $t = t_1$, $\vec{P}(t)$ is given by (6.23a). The first of these integrals is analytic and, since C_1 is a minimum point for $\phi = ct$ with respect to values on γ, exists for $t < t_1$ and $t > t_1$. The integral will be denoted by $\vec{R}(t-t_1)$. The second integral is zero for $t < t_1$ and, by considerations similar to what were employed to deduce (6.31) and (6.32) gives rise to an expression for $\vec{E}_0(t)$ of the form of (6.32). Hence in the neighborhood of $t = t_1$,

$$\begin{aligned} \vec{E}_0 &= \vec{R}(t-t_1) && \text{for } t < t_1 \\ &= \vec{R}(t-t_1) - \sqrt{t-t_1} \sum_{\nu} \frac{\vec{C}_{\nu}^{(1)}}{\nu!} (t-t_1)^{\nu} && \text{for } t > t_1 \end{aligned}$$

Also, the function $\vec{R}(t-t_1)$ is regular at $t = t_2$.

We now apply the general formula (2.6). The result is

$$\text{(6.35)} \qquad \begin{aligned} \vec{U}_1 &= -\sum_{0}^{\infty} \frac{1}{\nu!} \frac{\vec{C}_{\nu}^{(1)}}{(-i\omega)^{\nu + 1/2}} \Gamma(\nu + 3/2) \\ \vec{U}_2 &= -\sum_{0}^{\infty} \frac{1}{\nu!} \frac{\vec{C}_{\nu}^{(2)}}{(i\omega)^{\nu + 1/2}} \Gamma(\nu + 3/2) \; . \end{aligned}$$

The main terms of these series are

$$\vec{U}_1 \approx -\frac{3}{2}\sqrt{\pi}\,\frac{\vec{C}_0^{(1)}}{(-i\omega)^{1/2}} \quad ; \quad \vec{U}_2 \approx -\frac{3}{2}\sqrt{\pi}\,\frac{\vec{C}_0^{(2)}}{(i\omega)^{1/2}}$$

where $\vec{C}_0^{(1)}$ and $\vec{C}_0^{(2)}$ are given by (6.33). The formulae (6.35) demonstrate that $\vec{U}_1$, $\vec{U}_2$ and similarly $\vec{V}_1$, $\vec{V}_2$ approach zero if $\omega \to \infty$; i.e., if $\lambda \to 0$.

We know $\phi(x,y,z;p,q)$ has exactly one stationary value in D if x,y,z lies in the regular part of the field of rays. This value is either a maximum or a minimum. We have found that the part of the diffraction integrals (4.15) which is characterized by the stationary point approaches asymptotically the given small wave length field $\vec{A}\sqrt{K}\ e^{ik\psi}$, $\vec{B}\sqrt{K}\ e^{ik\psi}$ if $k \to \infty$. The remaining parts of the diffraction integral converge to zero. It follows that the solutions $\vec{u}, \vec{v}$ given by (4.15) have the asymptotical behavior which we have postulated originally.

In the irregular part of the field several interior stationary points of ϕ and more than 2 stationary points of the boundary values may be obtained. The contributions from these latter points converge to zero if $k \to \infty$. On the other hand the contributions from the interior stationary points give, for large values of k, the asymptotic fields

$$\vec{u} \approx \vec{A}'\sqrt{K'}\ e^{ik\psi'} + \vec{A}''\sqrt{K''}\ e^{ik\psi''} + \ldots$$

$$\vec{v} \approx \vec{B}'\sqrt{K'}\ e^{ik\psi'} + \vec{B}''\sqrt{K''}\ e^{ik\psi''} + \ldots$$

i.e., the asymptotic behavior required in (3.9). This is a consequence of the result of article 6.3 in the case of minimum points. The same consideration, of course, can be made in case of maximum points. Saddle points require a slight modification of the method ($\vec{P}_t(t)$ and $\vec{Q}_t(t)$ have logarithmic singularities) but the asymptotic result is the same as before.

We also mention the fact that our assumption about the analyticity of $\vec{A}(p,q)$ is not essential. If $\vec{A}(p,q)$ is sectionally analytic but discontinuous on certain curves in the domain D, then these curves have the same effect as the boundary curve Γ of D, that is, they introduce new points where the contour lines ϕ = ct are tangential to the boundary curve.

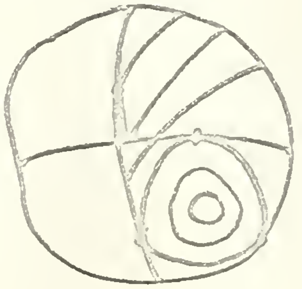

Figure 30

7. Asymptotic Behavior of the Diffraction Integrals at Infinity.

7.1 Articles 5 and 6 analyzed the asymptotic behavior of the diffraction integrals (4.15) for large k or small λ. It remains to be shown that the integrals (4.15) satisfy the boundary conditions at infinity as formulated in article 3.4. We do this simply by transforming this problem in such a manner that the results of the preceding article can be applied.

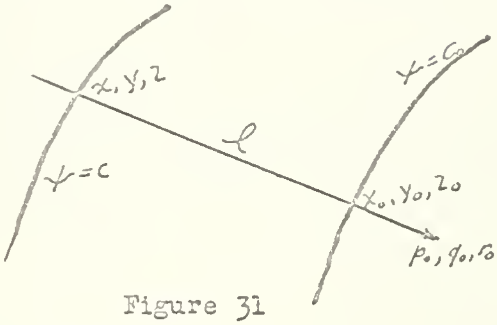

Figure 31

Consider a fixed point x_o, y_o, z_o and a ray p_o, q_o, r_o through this point. Let (x, y, z) be any point on this ray. x, y, and z are given by

$$(7.1)\qquad \begin{aligned} x &= x_o - \frac{\ell}{n}\, p_o \\ y &= y_o - \frac{\ell}{n}\, q_o \\ z &= z_o - \frac{\ell}{n}\, r_o \end{aligned}$$

For increasing ℓ, the point (x,y,z) travels on this ray twoards infinity.

We wish to determine the asymptotic behavior of the vectors $\vec{u}, \vec{v}$ for large values of ℓ. We consider only the vector

$$(7.2)\qquad \vec{u} = -\frac{ik}{2\pi} \iint \frac{\vec{A}(p,q)}{r}\, e^{ik(W+px+qy+rz)}\, dpdq\ .$$

which is one of the integrals (4.15). We introduce the formulas (7.1) into this integral and using the notation

$$(7.3)\qquad \vec{A}^{*} = n\vec{A}\, e^{ik(W+px_o + qy_o + rz_o)}\ .$$

it follows that

$$\ell\vec{u} = -\frac{ik\ell}{2\pi n} \iint \frac{\vec{A}^{*}(p,q)}{r}\, e^{-ik\frac{\ell}{n}(pp_o + qq_o + rr_o)}\, dpdq$$

or letting

$$k' = \frac{k\ell}{n}\ :$$

$$(7.4)\qquad \ell\vec{u} = -\frac{ik'}{2\pi} \iint \frac{\vec{A}^{*}}{r}\, e^{-ik'(pp_o + qq_o + rr_o)}$$

Comparison of this integral with the integral in (7.2) shows that (7.4) has the same mathematical form as the original diffraction integral. Its asymptotic behavior for large values of k' thus must be the same as that of the diffraction integrals (4.15) or (7.2) in case of large values of k. Letting

$$\phi = -(pp_0 + qq_0 + rr_0) \tag{7.5}$$

We recognize that ϕ has a minimum point at $p = p_0$, $q = q_0$, $r = r_0$; the minimum value being $\phi = -n^2$. The results of article 6, formulas (6.23) and (6.6) in particular, permit us to conclude that

$$\ell \vec{u} - \frac{\vec{A}^*(p_0,q_0)}{r_0} \frac{e^{ik'n^2}}{\sqrt{\phi_{pp}\,\phi_{qq} - \phi_{pq}^2}} \to 0 \tag{7.6}$$

if $\ell \to \infty$, i.e., if $k' \to \infty$. The quantity, $-n^2$, plays the role here that $\psi(x,y,z)$ plays as the minimum of ϕ in (7.2). Now, at $p = p_0$, $q = q_0$, we find from (7.5) that

$$\phi_{pp}\,\phi_{qq} - \phi_{pq}^2 = \frac{n^2}{r_0^2} \,. \tag{7.7}$$

Moreover,

$$\vec{A}^*(p_0,q_0)\, e^{ik'n^2} = n\,\vec{A}(p_0,q_0)\; e^{ik(W + p_0x_0 + q_0y_0 + r_0z_0 - \ell n)}$$

$$= n\,\vec{A}(p_0,q_0)\, e^{ik\psi(x,y,z)} ,$$

for, by formula (4.12)

$$W(p_0,q_0) + p_0x_0 + q_0y_0 + r_0z_0 = \psi(x_0,y_0,z_0)$$

and, since $\psi(x,y,z)$ is the optical path from the source to (x,y,z),

$$\psi(x_0,y_0,z_0) - \ell n = \psi(x,y,z) \,.$$

Hence it follows that

$$\ell \vec{u} - \vec{A}(p_0,q_0)\, e^{ik\psi} \to 0 \tag{7.8}$$

or, since $\ell\sqrt{K} \to 1$, that

$$\frac{\vec{u}}{\sqrt{K}}\, e^{-ik\psi} \to \vec{A}(p_0,q_0) \,. \tag{7.9}$$

This, however, is the condition (3.10).

The condition (3.13) for the derivatives of $\vec{u}$ and $\vec{v}$ can be derived in exactly the same manner.

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